

PERIODIC SCHUR PROCESS AND CYLINDRIC PARTITIONS

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ABSTRACT. Periodic Schur process is a generalization of the Schur process introduced in [OR1] (math.CO/0107056). We compute its correlation functions and their bulk scaling limits, and discuss several applications including asymptotic analysis of uniform measures on cylindric partitions, time-dependent extensions of the discrete sine kernel, and bulk limit behavior of certain measures on partitions introduced in [NO] (hep-th/0306238) in connection with supersymmetric gauge theories.

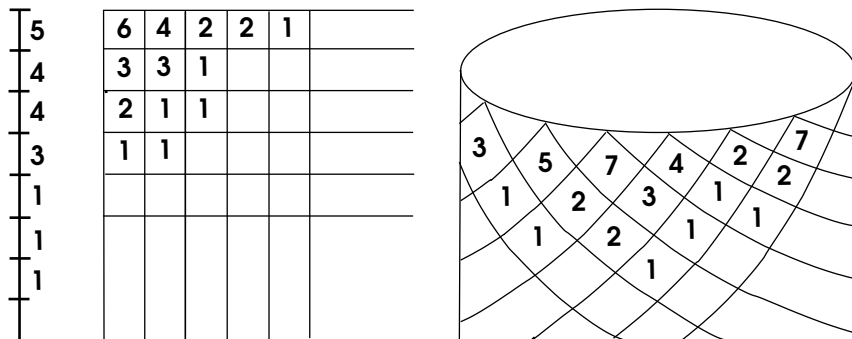
CONTENTS

Introduction

1. Periodic Schur process
 2. Correlation functions
 3. Bulk scaling limit
 4. Extensions of the discrete sine kernel
 5. Cylindric partitions
 6. Bulk of large cylindric partitions with finite or slowly growing period
 7. Bulk of large cylindric partitions with period of intermediate growth
 8. On a measure of Nekrasov and Okounkov
- References

INTRODUCTION

One way to see how the content of this paper is different from previous works on the subject is to examine the following three pictures.



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The leftmost one is a schematic image of an ordinary *partition* — a way of representing a natural number as an unordered sum of natural summands.¹ Partitions can also be viewed as ways of inscribing weakly decreasing nonnegative integers into unit intervals filling the half-line so that the total number of nonzero entries is finite.

The middle picture represents a *plane partition* — a way of filling the boxes of the square grid in the quarter plane with nonnegative integers so that the numbers do not increase as we move to infinity in the directions of the axes, and the total number of nonzero entries is also finite.

Finally, the rightmost picture represents a *cylindric partition* — a way of filling the boxes of the square grid wrapped around a half-cylinder with nonnegative integers so that the numbers do not increase as we move away from the border of the cylinder in either of the two perpendicular directions of the grid lines. The total number of nonzero entries is also required to be finite.

In this paper we initiate the study of *random* cylindric partitions.

Random (ordinary) partitions or, in other words, various probability measures on partitions, have been extensively studied since 1940's. The number of references is so large that we will not even attempt to list them. An excellent survey of uses of random partitions is available in [O2].

Random plane partitions are less common, partly because they are much harder to study. Substantial progress in understanding the uniform measure on plane partitions with given norm (=sum of filling numbers) was achieved only recently, see [CK], [OR1], [OR2]. In particular, the authors of [OR1] introduced new techniques which allowed them to derive determinantal formulas for the correlation functions of random plane partitions with weights proportional to q^{norm} , $0 < q < 1$.²

The main object of [OR1] called the *Schur process* is a generalization of an earlier concept called the *Schur measure* introduced in [O1] to handle certain measures on (ordinary) partitions. The range of applications of Schur measures and Schur processes, apart from uniform measures on plane partitions, is remarkably broad; examples include harmonic analysis on the infinite symmetric group [O1, §2.1.4], [BO1], Szegő-type formulas for Toeplitz determinants [BOK], relative Gromov-Witten theory of \mathbb{C}^* [OP], random domino tilings of the Aztec diamond [J2], discrete and continuous polynuclear growth processes in one space and one time dimensions [PS], [J1], etc.

In this paper we introduce and study a generalization of the Schur process which we call the *periodic Schur process*. We derive explicit formulas for the correlation functions of this new process and use them to compute the correlation functions of the random cylindric partitions with weights proportional to q^{norm} . This result allows us to obtain various (bulk) limits of these correlation functions as $q \rightarrow 1$. The limiting cases differ by how fast the radius of the cylinder grows comparing to $|\ln q|^{-1}$, and the results depend in a nontrivial way on the angle between the grid lines and the axis of the cylinder.

We also present two other applications of the periodic Schur process.

First, we use it to construct an infinite-dimensional family of determinantal point processes on \mathbb{Z}^2 which extend the well-known one-dimensional discrete sine process.

¹In this example the number 19 is represented as $19=5+4+4+3+1+1+1$.

²For such measures the conditional distribution of the plane partitions with fixed norm is always uniform (and independent of q). For this reason these measures are often also called uniform.

This family includes two previously obtained in [OR1] and [BO2] extensions as special cases. Such abundance of two-dimensional extensions is rather unexpected: In all previously known examples, probabilistic models yielded only one extension per model, and for most one-dimensional determinantal point processes no more than one extension is known.

The second application is a computation of the correlation functions and their bulk scaling limits for a measure on (ordinary) partitions introduced in [NO] in connection with certain supersymmetric gauge theories.

Let us describe our results in more detail.

The periodic Schur process depends on a natural number N (the period), a parameter t , $|t| < 1$, and $2N$ specializations $a[1], b[1], \dots, a[N], b[N]$ of the algebra Λ of symmetric functions.³ The process lives on periodic sequences of $2N$ partitions

$$\lambda^{(N)} = \lambda^{(0)} \supset \mu^{(1)} \subset \lambda^{(1)} \supset \dots \subset \lambda^{(N-1)} \supset \mu^{(N)} \subset \lambda^{(N)} = \lambda^{(0)},$$

and it assigns to such a sequence the weight proportional to

$$t^{|\lambda^{(0)}|} s_{\lambda^{(0)}/\mu^{(1)}}(a[1]) s_{\lambda^{(1)}/\mu^{(1)}}(b[1]) \cdots s_{\lambda^{(N-1)}/\mu^{(N)}}(a[N]) s_{\lambda^{(N)}/\mu^{(N)}}(b[N]).$$

Here $s_{\lambda/\mu}$'s are the skew Schur functions. The proportionality coefficient (which we explicitly compute) is chosen so that the sum of all weights is equal to 1. The weights can be viewed either as complex numbers or as formal series in $\Lambda^{\otimes 2N}[t]$.

When $t = 0$ the partition $\lambda^{(0)} = \lambda^{(N)}$ must be empty in order for the sequence to have a nonzero weight, and the periodic Schur process turns into the conventional Schur process of [OR1]. On the other hand, if all the specializations $a[k], b[k]$ are trivial, the periodic Schur process turns into the uniform measure on partitions, which assigns to a partition λ the weight proportional to $t^{|\lambda|}$.

Denote by \mathbb{Z}' the set $\mathbb{Z} + \frac{1}{2}$. To any sequence of partitions as above it is convenient to associate a point configuration (subset) in $\{1, \dots, N\} \times \mathbb{Z}' = \mathbb{Z}' \sqcup \dots \sqcup \mathbb{Z}'$ given by

$$\{\lambda_i^{(1)} - i + \frac{1}{2}\}_{i \geq 1} \sqcup \dots \sqcup \{\lambda_i^{(N)} - i + \frac{1}{2}\}_{i \geq 1}.$$

This set determines the sequence $(\lambda^{(1)}, \dots, \lambda^{(N)})$ uniquely.

Correlation functions are defined as probabilities, with respect to the periodic Schur process,⁴ that this random point configuration contains a fixed finite set of points:

$$\rho_n(\tau_1, x_1, \dots, \tau_n, x_n) = \text{Prob} \left\{ x_j \in \{\lambda_i^{(\tau_j)} - i + \frac{1}{2}\}_{i \geq 1} \mid j = 1, \dots, n \right\}.$$

It turns out that the algebraic structure of the correlation functions substantially simplifies if one considers a modification of the periodic Schur process which we call the *shift-mixed* periodic Schur process. It can be viewed as the product measure of the periodic Schur process and a measure on \mathbb{Z} given by

$$\text{Prob}\{S\} = \text{const} \cdot z^S t^{\frac{S^2}{2}}, \quad S \in \mathbb{Z}, \quad z \in \mathbb{C},$$

³A specialization of Λ is an algebra homomorphism of Λ to \mathbb{C} .

⁴One should remember that these “probabilities” do not have to be positive and, moreover, may be just formal series of symmetric functions.

mapped to the space of point configurations in $\{1, \dots, N\} \times \mathbb{Z}'$ via

$$(\lambda, S) \mapsto \{S + \lambda_i^{(1)} - i + \frac{1}{2}\}_{i \geq 1} \sqcup \dots \sqcup \{S + \lambda_i^{(N)} - i + \frac{1}{2}\}_{i \geq 1}.$$

In other words, all points of the random point configuration of the periodic Schur process are shifted by an independent integral valued random variable S distributed as above. Here z is a new complex parameter.

The normalization constant of the S -distribution is the inverse of one of the Jacobi theta-functions

$$\theta_3(z; t) = \sum_{S \in \mathbb{Z}} z^S t^{\frac{S^2}{2}} = \prod_{n \geq 1} (1 - t^n) \prod_{n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} (1 + t^n z)(1 + t^n / z),$$

see e.g. [Er, 13.19(16)]. We assume that $z \neq -t^{\pm \frac{1}{2}}, -t^{\pm \frac{3}{2}}, \dots$, so that $\theta_3(z; t) \neq 0$.

The correlation functions of the shift-mixed process are defined in the same way as those of the initial process, and we denote these new functions as ρ_n^{shift} . It is not hard to see that ρ_n is equal to the constant term in z of $\theta_3(z; t) \rho_n^{\text{shift}}$.

In order to state our first result, we need to introduce more notation. Set $a_m[k] := \frac{1}{m} p_m(a[k])$, $b_m[k] := \frac{1}{m} p_m(b[k])$, where p_k 's are the Newton power sums. For any $\tau = 1, \dots, N$ denote

$$F(\tau, \zeta) = \exp \sum_{n \geq 1} \left(\frac{\zeta^n}{1 - t^n} \sum_{k=1}^{\tau} b_n[k] + \frac{(t\zeta)^n}{1 - t^n} \sum_{k=\tau+1}^N b_n[k] - \frac{(t/\zeta)^n}{1 - t^n} \sum_{k=1}^{\tau} a_n[k] - \frac{(1/\zeta)^n}{1 - t^n} \sum_{k=\tau+1}^N a_n[k] \right).$$

Theorem A. *The correlation functions of the shift-mixed periodic Schur process have determinantal form: For any $n \geq 1$ and $(\tau_1, x_1), \dots, (\tau_n, x_n) \in \{1, \dots, N\} \times \mathbb{Z}'$ we have*

$$\rho_n^{\text{shift}}(\tau_1, x_1; \dots; \tau_n, x_n) = \det[K(\tau_i, x_i; \tau_j, x_j)]_{i,j=1}^n$$

where the generating series of the correlation kernel $K(\sigma, x; \tau, y)$ has the form

$$\sum_{x, y \in \mathbb{Z}'} K(\sigma, x; \tau, y) \zeta^x \eta^y = \begin{cases} \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + (z t^m)^{-1}}, & \sigma \leq \tau, \\ -\frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + z t^m}, & \sigma > \tau. \end{cases}$$

This statement can be understood in two different ways — as a formal identity of series in $\Lambda^{\otimes 2N}[t]$ or as a numeric equality under suitable convergence conditions.

Ramanujan's summation formula for ${}_1\psi_1$ -series shows that the two series in the formula above are expansions of one and the same holomorphic function in two disjoint annuli. This function can be expressed in terms of Jacobi theta-functions, see Remark 2.4 below for details.

For $t = 0$ Theorem A yields the determinantal formula for the correlation functions of the conventional Schur process initially proved in [OR1], see also [J1] and

[BR] for other proofs. Note that the argument presented in this paper provides an independent proof of this result.

Let us point out that Theorem A is not particularly trivial even in the simplest case of the uniform measure on partitions, which arises when all specializations $a[k]$ and $b[k]$ are trivial. Then one readily sees that off-diagonal values of the correlation kernel vanish, and the statement reduces to the fact that the shift-mixed version of the uniform measure on partitions is equivalent to the product of countably many independent Bernoulli measures, see Corollary 2.6 below. The author was not able to locate this fact in the literature, although certain formulas equivalent to it can be found in [O1].

The correlation functions of the initial periodic Schur process are *not* determinantal. Nevertheless, they possess a nice multivariate integral representation given in Corollary 2.8 below, apart from the fact that they can be extracted from the formula of Theorem A by taking the constant term in z as mentioned above.

The proof of Theorem A that we present in this paper is a verification rather than a derivation of the formula for the correlation functions. The initial proof involved the formalism of the Fock space and was similar in spirit to the derivations given in [O1] and [OR1]. However, we decided to leave it out of this paper because of its length and certain technical difficulties in justification of formal manipulations with operators in Fock spaces. As a matter of fact, our initial inspiration came from the work [Ts], where the *universal characters* – analogs of the Schur symmetric functions for nonpolynomial representations of the unitary groups – were represented as matrix elements of certain operators in Fock spaces. We hope to return to the Fock space formalism in a subsequent publication.

Our second result describes the “bulk limit” of the correlation functions of the periodic Schur process and its shift-mixed version as $t \rightarrow 1$ and the period N , as well as the specializations $a[k]$ and $b[k]$, remain fixed.

Theorem B. *Assume that $z \notin \mathbb{R}_{\leq 0}$; $a_m[k], b_m[k] = O(R^m)$ as $m \rightarrow \infty$ for some $0 < R < 1$ and all $k = 1, \dots, N$; and*

$$A_m := \sum_{k=1}^N a_m[k] = \sum_{k=1}^N \overline{b_m[k]} =: \overline{B}_m, \quad m = 1, 2, \dots$$

Then as $t \rightarrow 1$, the correlation functions of the periodic Schur process and its shift-mixed version have a limit in the following sense: Choose $x_1(t), \dots, x_n(t) \in \mathbb{Z}'$ such that $|\ln t| \cdot x_k(t) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j$ are independent of t . Then for any $1 \leq \tau_1, \dots, \tau_n \leq N$

$$\begin{aligned} \lim_{t \rightarrow 1} \rho_n^{\text{shift}}(\tau_1, x_1(t); \dots, \tau_n, x_n(t)) &= \det [\mathcal{K}_{\tau_i, \tau_j}^{(z, \gamma)}(x_i - x_j)]_{i, j=1}^n, \\ \lim_{t \rightarrow 1} \rho_n(\tau_1, x_1(t); \dots, \tau_n, x_n(t)) &= \det [\mathcal{K}_{\tau_i, \tau_j}^{(1, \gamma)}(x_i - x_j)]_{i, j=1}^n, \end{aligned}$$

where the limit correlation kernel has the form

$$\mathcal{K}_{\sigma, \tau}^{(z, \gamma)}(d) = \begin{cases} \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\exp\left(-\sum_{m \geq 1} \sum_{k=\sigma+1}^{\tau} (a_m[k]\zeta^{-m} + b_m[k]\zeta^m)\right)}{1+z^{-1} \exp\left(\gamma - \sum_{m \geq 1} (A_m \zeta^{-m} + B_m \zeta^m)\right)} \frac{d\zeta}{\zeta^{d+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\exp\left(\sum_{m \geq 1} \sum_{k=\tau+1}^{\sigma} (a_m[k]\zeta^{-m} + b_m[k]\zeta^m)\right)}{1+z \exp\left(-\gamma + \sum_{m \geq 1} (A_m \zeta^{-m} + B_m \zeta^m)\right)} \frac{d\zeta}{\zeta^{d+1}}, & \sigma > \tau. \end{cases}$$

Note that the limit correlation functions are invariant with respect to simultaneous shifts of the “space variables” x_i .

The assumption $A_m = \overline{B}_m$ is crucial here; without this condition being satisfied even the integrals above may become meaningless because the denominators would be allowed to vanish on the integration contours.

It is rather unusual that the limit can be computed in such generality.

Before proceeding to cylindric partitions, let us describe in more detail the two other applications of Theorems A and B mentioned earlier.

Recall that a classical theorem proved independently by Aissen–Edrei–Schoenberg–Whitney in 1951 [AESW], [Ed], and by Thoma in 1964 [Th], states that a sequence $\{c_n\}_{n=0}^\infty$, $c_0 = 1$, is totally positive⁵ if and only if its generating series has the form

$$\sum_{n=0}^{\infty} c_n u^n = e^{\gamma u} \frac{\prod_{i \geq 1} (1 + \beta_i u)}{\prod_{i \geq 1} (1 - \alpha_i u)} =: TP_{\alpha, \beta, \gamma}(u)$$

for certain nonnegative parameters $\{\alpha_i\}$, $\{\beta_i\}$ and γ such that $\sum_i (\alpha_i + \beta_i) < \infty$.

Corollary 1. *For any doubly infinite sequences of totally positive parameter sets $\{(\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)})\}_{k \in \mathbb{Z}}$ and $\{(\tilde{\alpha}^{(k)}, \tilde{\beta}^{(k)}, \tilde{\gamma}^{(k)})\}_{k \in \mathbb{Z}}$ satisfying the additional conditions*

$$\alpha_i^{(k)}, \tilde{\alpha}_i^{(k)} \leq 1, \quad \beta_i^{(k)}, \tilde{\beta}_i^{(k)} < 1, \quad i \geq 1, \quad k \in \mathbb{Z},$$

and any $c \in (0, \pi)$, there exists a determinantal point process⁶ on $\mathbb{Z} \times \mathbb{Z}$ with the correlation kernel

$$\mathcal{K}(\sigma, x; \tau, y) = \begin{cases} \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} \prod_{k=\sigma+1}^{\tau} \left(TP_{\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)}}(\zeta) TP_{\tilde{\alpha}^{(k)}, \tilde{\beta}^{(k)}, \tilde{\gamma}^{(k)}}(\zeta^{-1}) \right)^{-1} \frac{d\zeta}{\zeta^{x-y+1}} \\ - \frac{1}{2\pi i} \int_{e^{ic}}^{e^{-ic}} \prod_{k=\tau+1}^{\sigma} \left(TP_{\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)}}(\zeta) TP_{\tilde{\alpha}^{(k)}, \tilde{\beta}^{(k)}, \tilde{\gamma}^{(k)}}(\zeta^{-1}) \right) \frac{d\zeta}{\zeta^{x-y+1}} \end{cases}$$

where the first formula is used for $\sigma \leq \tau$, the second formula is used for $\sigma > \tau$, and both integrals are taken over positively oriented arches of the unit circle.

The equal time values of the kernel above are exactly those of the discrete sine kernel on \mathbb{Z} : For any $\tau \in \mathbb{Z}$

$$\mathcal{K}(\tau, x; \tau, y) = \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} \frac{d\zeta}{\zeta^{x-y+1}} = \frac{\sin(c(x-y))}{\pi(x-y)}, \quad x, y \in \mathbb{Z}.$$

Thus, the kernels $\mathcal{K}(\sigma, x; \tau, y)$ are extensions of the discrete sine kernel.

The choice of $\alpha_1^{(k)} \equiv 1$ and all other parameters being zero brings us to the incomplete beta kernel of [OR1]. On the other hand, taking $\gamma^{(k)} = \tilde{\gamma}^{(k)}$ with all other parameters being zero yields the extension of the discrete sine kernel obtained in [BO2, Theorem 4.2].

⁵By definition, this sequence is totally positive if all minors of the matrix $[c_{i-j}]_{i,j \geq 0}$ are nonnegative.

⁶This time all the probabilities are nonnegative!

As for the second application, we consider a probability measure on the set of all partitions given by the formula, see [NO, §6.2],

$$M_{\mu,t}(\lambda) = \prod_{n \geq 1} (1 - t^n)^{1-\mu^2} \cdot t^{|\lambda|} \prod_{\square \in \lambda} \frac{h(\square)^2 - \mu^2}{h(\square)^2}, \quad \lambda \in \mathbb{Y}.$$

Here $\mu \in i\mathbb{R}$ and $t \in (0, 1)$ are the parameters, the product is taken over all boxes of the Young diagram λ , and $h(\square)$ denotes the length of the hook rooted at the box \square .

One remarkable feature of this measure is that it interpolates between the uniform measure on partitions, which appears at $\mu = 0$, and the (poissonized) Plancherel measure on partitions (see, e.g., [BOO]), which is obtained from $M_{\mu,t}$ by the limit transition $t \rightarrow 0$, $\mu \rightarrow \pm i\infty$, $t|\mu|^2 \rightarrow \theta > 0$.

The measures $M_{\mu,t}$ may be viewed as special cases of the periodic Schur process with period $N = 1$, and Theorem B leads to the following statement.

Corollary 2. *As $t \rightarrow 1$, the correlation functions of $M_{\mu,t}$ have the following limit: Choose $x_1(t), \dots, x_n(t) \in \mathbb{Z}'$ as in Theorem B. Then the correlation functions converge to determinants of the limit correlation kernel*

$$\mathcal{K}^{(\gamma,\mu)}(x, y) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{1}{1 + e^\gamma(1 - \zeta)^{-\mu}(1 - \zeta^{-1})^\mu} \frac{d\zeta}{\zeta^{x-y+1}}, \quad x, y \in \mathbb{Z}.$$

Now let us return to cylindric partitions.

It is convenient for us to represent cylindric partitions as periodic sequences of ordinary partitions by reading the filling numbers along the diagonal rays which form 45° angle with grid lines. For example, the visible part of the cylindric partition represented by the picture in the beginning of this introduction gives the sequence of partitions

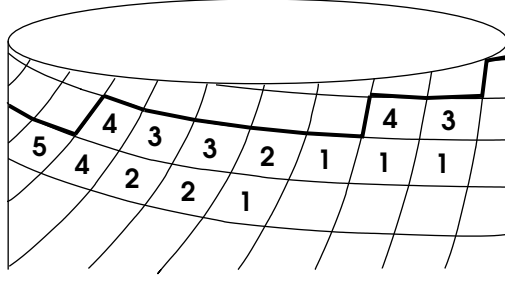
$$\dots (3) \supset (1) \subset (5, 1) \supset (2) \subset (7, 2) \supset (3, 1) \subset (4, 1) \supset (1) \subset (2, 1) \supset (2) \subset (7) \dots$$

The condition of filling numbers not increasing along the grid lines is equivalent to neighboring partitions in such a sequence having *interlacing parts*. It is also equivalent to saying that the Young diagrams of any pair of neighboring partitions are different by either adding or removing a horizontal strip. Such a relation between two partitions (κ, ν) is denoted as $\kappa \succ \nu$ or $\kappa \prec \nu$, depending on which of these two partitions is larger.

The choice of \succ 's and \prec 's between neighboring partitions is exactly the choice of the boundary profile of our cylindric partition near the cut of the cylinder. We will fix such a profile by providing two periodic sequences $\{A[k]\}$ and $\{B[k]\}$ of 0's and 1's such that $A[k] + B[k] \equiv 1$; the \succ 's correspond to $A[k] = 1$ and $B[k] = 0$, and \prec 's correspond to $A[k] = 0$, $B[k] = 1$. The ratio \varkappa of the total number of \prec 's in a period over the total number of \succ 's in a period will be called the *slope* of the profile. The slope depends only on the angle between the grid lines and the axis of the cylinder. The case $\varkappa = 1$ corresponds to the diagonal rays being parallel to the cylinder axis.

The following picture represents a cylindric partition with slope $\varkappa < 1$, and the visible part of the boundary profile corresponds to the sequences

$$\begin{aligned} \{A[k]\} &= (\dots, 1, 0, 1, 1, 1, 1, 0, 1, \dots), \\ \{B[k]\} &= (\dots, 0, 1, 0, 0, 0, 0, 1, 0, \dots). \end{aligned}$$



We will denote by N the period of the sequences above. For any integer m , let $m(N)$ be the smallest positive integer such that $m \equiv m(N) \pmod{N}$.

Proposition. *For any profile $\{A[k]\}_{k=1}^N, \{B[k]\}_{k=1}^N$ as above we have*

$$\sum_{\substack{\text{cylindric partitions } \pi \\ \text{of the given profile}}} s^{|\pi|} = \prod_{n \geq 1} \frac{1}{1 - s^{nN}} \prod_{\substack{p \in \overline{1, N}: A[p]=1 \\ q \in \overline{1, N}: B[q]=1}} \frac{1}{1 - s^{(p-q)(N) + (n-1)N}}.$$

Here $|\cdot|$ denotes the norm (=the sum of filling numbers) of cylindric partitions.

We were unable to find this formula in the literature, although it may well follow from more refined results of [GK]. In the limit when the radius of the cylinder becomes large, if $A = (1, \dots, 1, 0, \dots, 0)$, $B = (0, \dots, 0, 1, \dots, 1)$, and \varkappa remains bounded away from 0 and ∞ (this means that the boundary of the cylindric partitions locally looks like the boundary of the quarter plane), the formula of the proposition reproduces the celebrated formula of MacMahon for the sum of the weights s^{norm} over all plane partitions.

The applicability of the periodic Schur process to cylindric partitions follows from the following basic property of the skew Schur functions evaluated at a single indeterminate x : $s_{\kappa/\nu}(x) = x^{|\kappa| - |\nu|}$ if $\kappa \succ \nu$ and 0 otherwise. One readily checks that the periodic Schur process with $t = s^N$ and specializations $a[k]$ and $b[k]$ being the evaluations at $s^k A[k]$ and $s^{-k} B[k]$, respectively, for all $k = 1, \dots, N$, is exactly the measure $\text{const} \cdot s^{\text{norm}}$ on cylindric partitions with a fixed profile described by $\{A[k]\}$ and $\{B[k]\}$. From now on we will use the term “uniform measure” for this distribution.

The above observation implies that Theorem A provides a determinantal formula for correlation functions of the shift-mixed modification of the uniform measure on cylindric partitions.⁷ Our next goal is to explain what happens to Theorem B.

In order to state our next result, we need to introduce a curve Γ_\varkappa in the complex plane via (here \varkappa is a positive parameter)

$$\Gamma_\varkappa = \left\{ -\frac{\sin \frac{\varphi}{1+\varkappa}}{\sin \frac{\varphi}{1+\varkappa-1}} e^{i\varphi} \mid \varphi \in [-\pi, \pi] \right\}.$$

This is a piecewise smooth closed curve which has a corner-like singularity at the point 1. We orient Γ_\varkappa counterclockwise.

For $\sigma \leq \tau$ set $A(\sigma, \tau] = \sum_{k=\sigma+1}^{\tau} A[k]$, and similarly for $B(\sigma, \tau]$. The slope \varkappa is equal to $B(0, N]/A(0, N]$.

⁷By correlation functions of the uniform measure on cylindric partitions we mean the correlation functions of the corresponding periodic Schur process, and similarly for the shift-mixed versions.

Theorem C. *In the limit $s \rightarrow 1$, the correlation functions of the uniform measure on cylindric partitions with a given profile $\{A[k], B[k]\}_{k=1}^N$ have a limit in the following sense: Choose $x_1(s), \dots, x_n(s) \in \mathbb{Z} + \frac{1}{2}$ such that as $s \rightarrow 1$, $N|\ln s| \cdot x_k(s) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j$ are independent of s . Then for any $\tau_1, \dots, \tau_n \in \{1, \dots, N\}$*

$$\lim_{s \rightarrow 1^-} \rho_n(\tau_1, x_1(s); \dots, \tau_n, x_n(s)) = \det[\mathcal{K}_{\tau_i, \tau_j}^{(\gamma)}(x_i - x_j)]_{i,j=1}^n$$

where the correlation kernel has the form

$$\mathcal{K}_{\sigma, \tau}^{(\gamma)}(d) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\varkappa}} \frac{(1-\zeta)^{B(\sigma, \tau]}(1-\zeta^{-1})^{A(\sigma, \tau]}}{1 + e^{\gamma}(1-\zeta)^{B(0, N]}(1-\zeta^{-1})^{A(0, N]}} \frac{d\zeta}{\zeta^{d+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int_{\Gamma_{\varkappa}} \frac{(1-\zeta)^{-B(\tau, \sigma]}(1-\zeta^{-1})^{-A(\tau, \sigma]}}{1 + e^{-\gamma}(1-\zeta)^{-B(0, N]}(1-\zeta^{-1})^{-A(0, N]}} \frac{d\zeta}{\zeta^{d+1}}, & \sigma > \tau. \end{cases}$$

The function $(1-\zeta)^{B(0, N]}(1-\zeta^{-1})^{A(0, N]}$ takes nonnegative values on Γ_{\varkappa} , and thus the integrals are correctly defined.

The limit density function $\mathcal{K}_{\tau, \tau}^{(\gamma)}(0)$ does not depend on τ , which means that it is invariant with respect to rotations of the cylindric partitions. Note also that it depends on the profile only through $A(0, N]$ and $B(0, N]$, or, in other words, through the period N and the slope \varkappa .

Interestingly enough, a formal application of Theorem B to the case of cylindric partitions may produce an incorrect answer if $\varkappa \neq 1$. This happens because for $\varkappa \neq 1$ the condition $A_m = \overline{B}_m$ of Theorem B is violated, and we need to deform the integration contours in order to perform the asymptotic analysis.

In Theorem C we kept the period N finite while sending s to 1. The next level of difficulty is to consider the case of periods growing together with $|\ln s|^{-1}$.

If the growth of the period is slow in the sense that $N|\ln s|$ still tends to 0, then we prove that the limit behavior of the correlation functions can be read off Theorem C above by taking the limit $N \rightarrow \infty$ and keeping the slope \varkappa fixed. In the limit one sees extensions of the discrete sine kernel as in Corollary 1 above with the parameters $\alpha_1^{(k)}$ or $\tilde{\alpha}_1^{(k)}$ taking values between 0 and 1 and all other parameters being zero. Details can be found in §6.

The case of the period N growing in such a way that the product $N|\ln s|$ has a finite limit, is substantially more complicated. The reason is simple – the limiting behavior depends on the details of the profile rather than just on its slope. In this paper we only consider the case of the slope being equal to 1, and we prove two results.

First, we show that if the profile sequences $\{A[k]\}$ and $\{B[k]\}$ are periodic with a finite period (in addition to being periodic with the growing period N) then the limit behavior is just the same as in the case of the slowly growing periods.

Second, we consider the corner-like profiles with $\{A[k]\}$ and $\{B[k]\}$ consisting of one block of 0's and one block of 1's. We compute the limit of the correlation functions near two “corners”, where the partitions are the largest and the smallest. The results in both cases are governed by the incomplete beta kernel with the density functions given by certain analytic expressions involving elliptic functions. Details can be found in §7.

To conclude the introduction, let us mention two circles of questions which we do not discuss in this paper, but which are certainly very interesting.

The existence of limit correlation functions which decay fast enough when the distance between the arguments grows, is a strong indication for the existence of a *limit shape* of the corresponding random (ordinary, plane or cylindric) partitions. Such a decay is present in all the cases we considered. The limit density function allows one to predict what the limit shape would look like (see, for example, Comment 3 after Theorem 3.1 below), but one needs additional arguments to actually prove the concentration phenomenon. For example, for the measures $M_{\mu,t}$ considered in Corollary 2 above (and, in fact, for a substantially larger class of measures on partitions), the existence of the limit shape as $\mu \rightarrow \pm i\infty$ was proved in [NO] by variational techniques.

Also observe that in this paper we consider only what is usually called “the bulk scaling limit” of the correlation functions. It would be very interesting to study the “edge scaling limit” as well. In the case of the uniform measure on (skew) plane partitions different edge scaling limits were computed in [OR2].

Acknowledgements. The author is very grateful to Grigori Olshanski for numerous discussions which were extremely useful and stimulating. In particular, a large part of §8 below consists of results of such discussions. The author would also like to thank Eric Rains for pointing out a Cauchy type determinant formula for theta functions due to Frobenius; it is used in the proof of Corollary 2.8 below.

This research was partially supported by the NSF grant DMS-0402047 and the CRDF grant RIM1-2622-ST-04.

1. PERIODIC SCHUR PROCESS

Fix a natural number N and consider a measure on periodic sequences of $2N$ partitions

$$\lambda^{(N)} = \lambda^{(0)} \supset \mu^{(1)} \subset \lambda^{(1)} \supset \dots \subset \lambda^{(N-1)} \supset \mu^{(N)} \subset \lambda^{(N)} = \lambda^{(0)}$$

by specifying the weight of such a sequence to be equal to

$$W(\lambda, \mu) = t^{|\lambda^{(0)}|} s_{\lambda^{(0)}/\mu^{(1)}}(a[1]) s_{\lambda^{(1)}/\mu^{(1)}}(b[1]) \cdots s_{\lambda^{(N-1)}/\mu^{(N)}}(a[N]) s_{\lambda^{(N)}/\mu^{(N)}}(b[N]).$$

Here t is a parameter and $a[m], b[m]$, $m = 1, \dots, N$, are arbitrary specializations of the algebra Λ of symmetric functions.⁸ We will use the notation

$$a_k[m] := \frac{1}{k} p_k(a[m]), \quad b_k[m] := \frac{1}{k} p_k(b[m]),$$

where p_k 's are the Newton power sums.

The weights $W(\lambda, \mu)$ may also be viewed as elements of $\Lambda^{\otimes 2N}[t]$; in that case the notation $f(a[k])$ or $f(b[k])$ for $f \in \Lambda$ just indicates to which of the copies of Λ in $\Lambda^{\otimes 2N}$ this symmetric function belongs.

The partition function of such a measure will be denoted as Z (the notation \mathbb{Y} below stands for the set of all partitions including the empty one)

$$Z(N, t, a, b) := \sum_{\lambda^{(1)}, \mu^{(1)}, \dots, \lambda^{(N)}, \mu^{(N)} \in \mathbb{Y}} W(\lambda, \mu).$$

⁸A specialization of Λ is an algebra homomorphism of Λ to \mathbb{C} .

We will use the term *periodic Schur process* for this measure.

It is clear that without loss of generality we may consider only the distribution of λ 's. Indeed, by making some of the specializations $a[k], b[k]$ trivial⁹ we can force any given $\mu^{(\cdot)}$ to coincide with a neighboring $\lambda^{(\cdot)}$.

The periodic Schur process as defined above is not symmetric with respect to the circular shifts $\lambda^{(k)} \mapsto \lambda^{(k+l \bmod N)}$. This lack of symmetry can be easily repaired. Introduce new specializations $\tilde{a}[m], \tilde{b}[m]$ of Λ by assigning the following values to the power sums:

$$\tilde{a}_k[m] := s^{-km} a_k[m], \quad \tilde{b}_k[m] := s^{km} b_k[m],$$

where $s^N = t$. Then we obtain

$$\begin{aligned} W(\lambda, \mu) &= s^{|\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(N)}|} \\ &\times s_{\lambda^{(0)}/\mu^{(1)}}(\tilde{a}[1]) s_{\lambda^{(1)}/\mu^{(1)}}(\tilde{b}[1]) \cdots s_{\lambda^{(N-1)}/\mu^{(N)}}(\tilde{a}[N]) s_{\lambda^{(N)}/\mu^{(N)}}(\tilde{b}[N]), \end{aligned}$$

and this expression already possesses the rotational symmetry. The index m of $\tilde{a}[m]$ and $\tilde{b}[m]$ can now be viewed as an element of $\mathbb{Z}/N\mathbb{Z}$. We will use both this form of the measure and the initial non-symmetric one.

When all the specializations are trivial the measure concentrates on the sequences with coinciding terms: all λ 's and μ 's become equal. The resulting distribution on one copy of \mathbb{Y} is the so-called uniform measure: $W(\lambda) = t^{|\lambda|}$. This observation shows, in particular, that if one sets $t = 1$ then the partition function may become infinite.

If $t = 0$ then $\lambda^{(0)} = \lambda^{(N)}$ must be empty, and the periodic Schur process coincides with the conventional Schur process introduced in [OR1], see also [BR].

In general, the periodic Schur process may be viewed as the conventional Schur process on \mathbb{Y} started at the uniform distribution instead of the empty partition, and conditioned to yield periodic trajectories.

When the period N is equal to 2, the periodic Schur process can be viewed as an analog of the Schur measure of [O1] for the universal characters of the unitary groups (see [Ko] for details on universal characters):

$$\sum_{\mu^{(1)}, \mu^{(2)} \in \mathbb{Y}} W(\lambda^{(1)}, \lambda^{(2)}; \mu^{(1)}, \mu^{(2)}) = S_{[\lambda^{(1)}, \lambda^{(2)}]'}(a[1], b[1]^-) S_{[\lambda^{(1)}, \lambda^{(2)}]'}(a[2]^-, b[2]).$$

Here the superscript “ $-$ ” in a specialization c stands for the change of signs of all the power sums: $p_k(c^-) := -p_k(c)$.

Introduce the notation

$$A_k := \sum_{m=1}^N a_k[m], \quad B_k := \sum_{m=1}^N b_k[m].$$

⁹The trivial specialization of Λ is characterized by the fact that all power sums p_k , $k \geq 1$, specialize to zero.

Proposition 1.1. *The partition function of the periodic Schur process has the form*

$$\begin{aligned} Z(N, t, a, b) &= \prod_{n \geq 1} \frac{1}{1 - t^n} \exp \left(\sum_{n=1}^{\infty} n \left(\sum_{1 \leq l < k \leq N} a_n[k] b_n[l] + \frac{t^n A_n B_n}{1 - t^n} \right) \right) \\ &= \prod_{n \geq 1} \frac{1}{1 - s^{nN}} \exp \left(\sum_{n=1}^{\infty} \frac{n}{1 - s^{nN}} \sum_{k, l=1}^N \left(\mathbf{1}_{k > l} s^{(k-l)n} + \mathbf{1}_{k \leq l} s^{(N+k-l)n} \right) \tilde{a}_n[k] \tilde{b}_n[l] \right). \end{aligned}$$

Remark 1.2. The equality above can be viewed either as an identity of formal series in $\Lambda^{\otimes 2N}[t]$ or as a numeric equality under the assumption that $|t| < 1$ and the series $\sum_{n \geq 1} n a_n[k] b_n[l]$ are absolutely convergent for any $k, l = 1, \dots, N$.

Remark 1.3. Here is a different way to write the formula for the partition function. For two specializations α and β of Λ set

$$H(\alpha; \beta) = \sum_{\lambda \in \mathbb{Y}} s_{\lambda}(\alpha) s_{\lambda}(\beta).$$

If α and β are evaluations of symmetric functions at variables (α_n) and (β_n) then the Cauchy identity, see e.g. [Macd, §I (4.3)], reads $H(\alpha; \beta) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1}$. It is also not hard to verify that $H(\alpha; \beta) = \exp \sum_{n \geq 1} p_n(\alpha) p_n(\beta) / n$. Hence,

$$\begin{aligned} Z(N, t, a, b) &= \prod_{n \geq 1} \frac{1}{1 - t^n} \prod_{1 \leq l < k \leq N} H(a[k]; b[l]) \prod_{n \geq 1} \prod_{k, l=1}^N H(t^n a[k]; b[l]) \\ &= \prod_{n \geq 1} \left(\frac{1}{1 - s^{nN}} \prod_{k > l} H(s^{(k-l)n} \tilde{a}[k]; \tilde{b}[l]) \prod_{k \leq l} H(s^{(N+k-l)n} \tilde{a}[k]; \tilde{b}[l]) \right), \end{aligned}$$

where the multiplication of a specialization α by a scalar q is defined via

$$p_n(q \cdot \alpha) := q^n p_n(\alpha), \quad n \geq 1.$$

Proof of Proposition 1.1. The argument uses the same idea as [Macd, Ex. I.5.28]. It is more convenient to work in terms of s, \tilde{a}, \tilde{b} . We have

$$\begin{aligned} Z(N, t, a, b) &= \sum_{\lambda, \mu} s^{|\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(N)}|} \\ &\quad \times s_{\lambda^{(N)}/\mu^{(1)}}(\tilde{a}[1]) s_{\lambda^{(1)}/\mu^{(1)}}(\tilde{b}[1]) \cdots s_{\lambda^{(N-1)}/\mu^{(N)}}(\tilde{a}[N]) s_{\lambda^{(N)}/\mu^{(N)}}(\tilde{b}[N]) \\ &= \sum_{\lambda, \mu} s^{|\mu^{(1)}| + |\mu^{(2)}| + \dots + |\mu^{(N)}|} \\ &\quad \times s_{\lambda^{(N)}/\mu^{(1)}}(s\tilde{a}[1]) s_{\lambda^{(1)}/\mu^{(1)}}(\tilde{b}[1]) \cdots s_{\lambda^{(N-1)}/\mu^{(N)}}(s\tilde{a}[N]) s_{\lambda^{(N)}/\mu^{(N)}}(\tilde{b}[N]) \\ &= H(s\tilde{a}[1]; \tilde{b}[N]) H(s\tilde{a}[2]; \tilde{b}[1]) \cdots H(s\tilde{a}[N]; \tilde{b}[N-1]) \sum_{\kappa, \mu} s^{|\mu^{(1)}| + |\mu^{(2)}| + \dots + |\mu^{(N)}|} \\ &\quad \times s_{\mu^{(1)}/\kappa^{(N)}}(\tilde{b}[N]) s_{\mu^{(1)}/\kappa^{(1)}}(s\tilde{a}[2]) \cdots s_{\mu^{(N)}/\kappa^{(N-1)}}(\tilde{b}[N-1]) s_{\mu^{(N)}/\kappa^{(N)}}(s\tilde{a}[1]) \end{aligned}$$

where to sum over λ 's we used the well-known formula

$$\sum_{\lambda} s_{\lambda/\mu}(\alpha) s_{\lambda/\hat{\mu}}(\beta) = H(\alpha; \beta) \sum_{\kappa} s_{\mu/\kappa}(\beta) s_{\hat{\mu}/\kappa}(\alpha),$$

see [Macd, Ex. I.5.26]. Applying the same trick to sum over μ 's we get

$$\begin{aligned} Z(N, t, a, b) &= H(s\tilde{a}[1]; \tilde{b}[N]) H(s\tilde{a}[2]; \tilde{b}[1]) \dots H(s\tilde{a}[N]; \tilde{b}[N-1]) \\ &\quad \times H(s^2\tilde{a}[1]; \tilde{b}[N-1]) H(s^2\tilde{a}[2]; \tilde{b}[N]) \dots H(s^2\tilde{a}[N]; \tilde{b}[N-2]) \\ &\times \sum_{\kappa, \rho} s^{|\kappa^{(1)}| + |\kappa^{(2)}| + \dots + |\kappa^{(N)}|} \\ &\quad \times s_{\kappa^{(N)}/\rho^{(1)}}(s^2\tilde{a}[2]) s_{\kappa^{(1)}/\rho^{(1)}}(\tilde{b}[N]) \dots s_{\kappa^{(N-1)}/\rho^{(N)}}(s^2\tilde{a}[1]) s_{\kappa^{(N)}/\rho^{(N)}}(\tilde{b}[N-1]). \end{aligned}$$

Continuing in the same fashion after N summations we obtain

$$Z(N, t, a, b) = \prod_{k>l} H(s^{k-l}\tilde{a}[k]; \tilde{b}[l]) \prod_{k\leq l} H(s^{N+k-l}\tilde{a}[k]; \tilde{b}[l]) \cdot Z(N, t, s^N a, b).$$

Iterating the above procedure we arrive at the final formula noting that both in the formal and analytic settings

$$\lim_{n \rightarrow \infty} Z(N, t, t^n a, b) = Z(N, t, \text{trivial}, b) = \prod_{n \geq 1} \frac{1}{1 - t^n}. \quad \square$$

Remark 1.4. Using the formula

$$\begin{aligned} \sum_{\lambda, \mu, \nu} s_{\kappa/\mu}(\alpha) s_{\lambda/\mu}(\beta) s_{\lambda/\nu}(\gamma) s_{\sigma/\nu}(\delta) &= H(\beta; \gamma) \sum_{\mu, \nu, \rho} s_{\kappa/\mu}(\alpha) s_{\mu/\rho}(\gamma) s_{\nu/\rho}(\beta) s_{\sigma/\nu}(\delta) \\ &= H(\beta; \gamma) \sum_{\rho} s_{\kappa/\rho}(\alpha, \gamma) s_{\sigma/\rho}(\beta, \delta) \end{aligned}$$

one readily sees that the projection of the periodic Schur process to

$$\lambda^{(0)}, \lambda^{(k_1)}, \dots, \lambda^{(k_{M-1})}, \lambda^{(k_M)} = \lambda^{(N)}$$

is again a periodic Schur process with a shorter period M and modified specializations

$$\begin{aligned} \hat{a}_n[1] &= \sum_{i=1}^{k_1} a_n[i], \quad \hat{a}_n[2] = \sum_{i=k_1+1}^{k_2} a_n[i], \quad \dots, \quad \hat{a}_n[M] = \sum_{i=k_{M-1}+1}^N a_n[i], \\ \hat{b}_n[1] &= \sum_{i=1}^{k_1} b_n[i], \quad \hat{b}_n[2] = \sum_{i=k_1+1}^{k_2} b_n[i], \quad \dots, \quad \hat{b}_n[M] = \sum_{i=k_{M-1}+1}^N b_n[i]. \end{aligned}$$

This fact can be used to define periodic Schur processes with *continuous time*.

2. CORRELATION FUNCTIONS

For any $\lambda \in \mathbb{Y}$ set

$$\mathcal{L}(\lambda) = \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1} \subset \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}.$$

By definition, the n th dynamical correlation function of the periodic Schur process is the probability that the random point configurations $\mathcal{L}(\lambda^{(\tau_k)})$ contain some fixed points $x_k \in \mathbb{Z}'$ for all $k = 1, \dots, n$:

$$\rho_n(\tau_1, x_1; \dots; \tau_n, x_n) = \frac{1}{Z(N, t, a, b)} \sum_{(\lambda, \mu) : x_k \in \mathcal{L}(\lambda^{(\tau_k)}), k=1, \dots, n} W(\lambda, \mu).$$

Some of the time moments τ_k may coincide, but if $(\tau_i, x_i) = (\tau_j, x_j)$ for $i \neq j$ then the correlation function vanishes.

In order to compute the correlation functions it is convenient to introduce a modification of the periodic Schur process which we call the *shift-mixed* periodic Schur process.

The shift-mixed process is a measure on point configurations in the disjoint union of N copies of \mathbb{Z}' obtained as follows. Let us take in the i th copy of \mathbb{Z}' the point configuration $\mathcal{L}(\lambda^{(i)})$, where $(\lambda^{(1)}, \dots, \lambda^{(N)})$ form the periodic Schur process, and let us shift all these N point configurations simultaneously by a random integer S distributed according to

$$\text{Prob}\{S\} = z^S t^{\frac{S^2}{2}} / \theta_3(z; t).$$

Here $z \neq 0$ is a new parameter, and $\theta_3(z; t) = \sum_{S \in \mathbb{Z}} z^S t^{\frac{S^2}{2}}$ is the partition function of the weights $z^S t^{\frac{S^2}{2}}$ which also happens to be one of the Jacobi theta-functions, see e.g. [Er, 13.19].

To summarize, in the shift-mixed periodic Schur process the weight of the point configuration of the form

$$\{S + \lambda_i^{(1)} - i + \frac{1}{2}\}_{i \geq 1} \sqcup \dots \sqcup \{S + \lambda_i^{(N)} - i + \frac{1}{2}\}_{i \geq 1}$$

is equal to $z^S t^{\frac{S^2}{2}} \sum_{\mu} W(\lambda, \mu)$, all point configurations not of this form have weight zero, and the partition function is equal $\theta_3(z; t) Z(N, t, a, b)$.

Since

$$\theta_3(z; t) = \prod_{n \geq 1} (1 - t^n) \cdot \prod_{n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} (1 + t^n z)(1 + t^n / z),$$

see e.g. [Er, 13.19(16)], we will always assume that $z \neq -t^{\pm \frac{1}{2}}, -t^{\pm \frac{3}{2}}, \dots$, so that the partition function is never zero.

The dynamical correlation functions of the shift-mixed process are defined in the same way as those of the initial process; we will denote them by ρ_n^{shift} .

Proposition 2.1. *The dynamical correlation functions of the periodic Schur process and its shift-mixed modification are related as follows:*

$$\begin{aligned} \rho_n(\tau_1, x_1; \dots; \tau_n, x_n) &= \text{constant term in } z \text{ of } \{ \theta_3(z; t) \rho_n^{\text{shift}}(\tau_1, x_1; \dots; \tau_n, x_n) \}, \\ \rho_n^{\text{shift}}(\tau_1, x_1; \dots; \tau_n, x_n) &= \frac{1}{\theta_3(z; t)} \sum_{S \in \mathbb{Z}} z^S t^{\frac{S^2}{2}} \rho_n(\tau_1, x_1 - S; \dots; \tau_n, x_n - S). \end{aligned}$$

Proof. Follows directly from the definition of the shift-mixed process. \square

The reason for introducing the shift-mixed process is the fact that this process is *determinantal*, i.e., its correlation functions can be written as certain determinants. To make an exact statement we need additional notation.

For any $\tau = 1, \dots, N$ set

$$\begin{aligned} F(\tau, \zeta) &= \exp \sum_{n \geq 1} \left(\frac{\zeta^n}{1 - t^n} \sum_{k=1}^{\tau} b_n[k] + \frac{(t\zeta)^n}{1 - t^n} \sum_{k=\tau+1}^N b_n[k] \right. \\ &\quad \left. - \frac{(t/\zeta)^n}{1 - t^n} \sum_{k=1}^{\tau} a_n[k] - \frac{(1/\zeta)^n}{1 - t^n} \sum_{k=\tau+1}^N a_n[k] \right) \\ &= \exp \sum_{n \geq 1} \left(\frac{(s^N \zeta)^n}{1 - s^{nN}} \sum_{k=\tau+1}^{N+\tau} \tilde{b}_n[k] s^{-kn} - \frac{(1/\zeta)^n}{1 - s^{nN}} \sum_{k=\tau+1}^{N+\tau} \tilde{a}_n[k] s^{kn} \right). \end{aligned}$$

Theorem 2.2. *The dynamical correlation functions of the shift-mixed periodic Schur process have determinantal form*

$$\rho_n^{\text{shift}}(\tau_1, x_1; \dots; \tau_n, x_n) = \det[K(\tau_i, x_i; \tau_j, x_j)]_{i,j=1}^n$$

where the generating series of the correlation kernel $K(\sigma, x; \tau, y)$ has the form

$$\sum_{x, y \in \mathbb{Z}'} K(\sigma, x; \tau, y) \zeta^x \eta^y = \begin{cases} \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + (zt^m)^{-1}}, & \sigma \leq \tau, \\ -\frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + zt^m}, & \sigma > \tau. \end{cases}$$

Remark 2.3. Similarly to Proposition 1.1, the formula above carries two statements: One holds in the algebra of formal series in $\Lambda^{\otimes 2N}[t]$ with z being an arbitrary nonzero complex number. To decompose the right-hand side into a series in t one uses the expansions

$$\frac{1}{1 + zt^m} = \begin{cases} \sum_{i \geq 0} (-zt^m)^i, & m > 0, \\ -\sum_{i \geq 1} (-zt^m)^{-i}, & m < 0, \end{cases} \quad \frac{1}{1 + (zt^m)^{-1}} = 1 - \frac{1}{1 + zt^m}.$$

The other statement is a numeric equality which we prove under the following convergence conditions:

$$\begin{aligned} a_n[k], b_n[l] &= O(R^n) \text{ for some } 0 < R < 1 \text{ and all } 1 \leq k, l \leq N; \\ |t| &< 1; \quad z \neq 0, -t^{\pm \frac{1}{2}}, -t^{\pm \frac{3}{2}}, \dots \end{aligned}$$

These conditions guarantee that the generating function above is an analytic function in ζ and η varying in an annulus either slightly outside or slightly inside the unit circle (depending on whether $\sigma \leq \tau$ or $\sigma > \tau$), and the values of the kernel are obtained as the Laurent coefficients of this function; see also the next remark.

Remark 2.4. The extraction of coefficients of the generating series can be performed by computing the corresponding contour integrals. Also, both series in the formula for the kernel above sum up to one and the same analytic function in disjoint domains:

$$\prod_{n \geq 1} (1 - t^n)^3 \frac{-\sqrt{y} \theta_3(yz; t)}{\theta_3(-yt^{-\frac{1}{2}}; t) \theta_3(z; t)} = \begin{cases} \sum_{m \in \mathbb{Z}'} \frac{y^m}{1 + (zt^m)^{-1}}, & 1 < |y| < |t|^{-1}, \\ - \sum_{m \in \mathbb{Z}'} \frac{y^m}{1 + zt^m}, & |t| < |y| < 1. \end{cases}$$

These are special cases of Ramanujan's summation formula for ${}_1\psi_1$ -series, see e.g. [GR, §5.2]. Indeed, the second sum equals

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \frac{y^{m+\frac{1}{2}}}{1 + zt^{m+\frac{1}{2}}} &= \frac{\sqrt{y}}{1 + zt^{\frac{1}{2}}} \sum_{m \in \mathbb{Z}} \frac{(-zt^{\frac{1}{2}}; t)_m y^m}{(-zt^{\frac{3}{2}}; t)_m} = \frac{\sqrt{y}}{1 + zt^{\frac{1}{2}}} {}_1\psi_1(-zt^{\frac{1}{2}}; -zt^{\frac{3}{2}}; t, y) \\ &= \frac{\sqrt{y}}{1 + zt^{\frac{1}{2}}} \frac{(t, t, -t^{\frac{1}{2}}zy, -t^{\frac{1}{2}}/(zy); t)_\infty}{(-t^{\frac{3}{2}}z, -t^{\frac{1}{2}}/z, y, t/y; t)_\infty} = \frac{(t; t)_\infty^3 \sqrt{y} \theta_3(yz; t)}{\theta_3(-yt^{-\frac{1}{2}}; t) \theta_3(z; t)} \end{aligned}$$

where we use the conventional notation

$$(a; t)_m = \prod_{n=0}^{m-1} (1 - at^n), \quad (a; t)_\infty = \prod_{n=0}^{\infty} (1 - at^n),$$

The first sum is obtained by the change $(y, z) \mapsto (y^{-1}, z^{-1})$.

Hence, in the analytic setting the formula for the correlation kernel above can be rewritten as follows:

$$K(\sigma, x; \tau, y) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)^2} \oint_{\zeta} \oint_{\eta} \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \frac{\theta_3(z\zeta\eta; t)}{\theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)} \frac{d\zeta d\eta}{\zeta^{x+\frac{1}{2}} \eta^{y+\frac{1}{2}}},$$

where both integration contours are simple loops going around the origin in positive direction such that $R < |\zeta|, |\eta| < R^{-1}$, and for $\sigma \leq \tau$ we have $1 < |\zeta\eta| < \min\{R^{-1}, |t|^{-1}\}$, while for $\sigma > \tau$ we have $\max\{R, |t|\} < |\zeta\eta| < 1$.

Remark 2.5. In the limit $t \rightarrow 0$ both the periodic Schur process and its shift-mixed version turn into the conventional Schur process of [OR1]. Accordingly, Theorem 2.2 yields a determinantal formula for the correlation functions of the Schur process derived in [OR1] (see also [J1] and [BR] for different proofs). Note that the summation formulas of Remark 2.4 just turn into geometric series

$$\frac{\sqrt{y}}{y-1} = \begin{cases} \sum_{m \in \mathbb{Z}'_-} y^m, & |y| > 1, \\ - \sum_{m \in \mathbb{Z}'_+} y^m, & |y| < 1. \end{cases}$$

Corollary 2.6. *The shift-mixed uniform measure on partitions is isomorphic to the product of independent Bernoulli random variables on $\{0, 1\}^{\mathbb{Z}'}$ with*

$$\text{Prob}\{x_m = 1\} = \frac{zt^m}{1 + zt^m}, \quad m \in \mathbb{Z}',$$

in the following sense: Pairs $(\lambda, S) \in \mathbb{Y} \times \mathbb{Z}$ are in one-to-one correspondence with the sequences $(x_m) \in \{0, 1\}^{\mathbb{Z}'}$ of nonzero weights via

$$x_m = \begin{cases} 1, & \text{if } m \in \{S + \lambda_i - i + \frac{1}{2}\}, \\ 0, & \text{otherwise,} \end{cases}$$

and the weight of such a sequence is equal to

$$\text{Prob}\{\lambda, S\} = \frac{t^{|\lambda| + \frac{S^2}{2}} z^S}{\prod_{n=\frac{1}{2}, \frac{3}{2}, \dots} (1 + t^n z)(1 + t^n/z)}.$$

Proof. Follows from the fact that for the trivial specializations a and b the periodic Schur process turns into the uniform measure (see §1), the function $F(\cdot, \cdot)$ becomes identically equal to 1, and the equal time values of the correlation kernel are readily seen to be equal to

$$K(\tau, x; \tau, y) = \frac{zt^x}{1 + zt^x} \delta_{x,y}. \quad \square$$

Remark 2.7. Corollary 2.6 is equivalent to [O1, (3.14)]. It is also fairly easy to prove this statement independently by explicitly computing the weight of a sequence in $\{0, 1\}^{\mathbb{Z}'}$. This will essentially be done in the proof of Proposition 2.12 below.

Before proceeding to the proof of Theorem 2.2 let us draw one more corollary. Proposition 2.1 explains that one can obtain the correlation functions of the periodic Schur process by extracting the constant coefficient in z from the determinantal formula of Theorem 2.2. In fact, this extraction can be performed explicitly yielding a multivariate integral representation. To state the result it is more convenient to work with another Jacobi theta function $\theta_1(x; t)$ defined as follows, cf. [Er, 13.19]:

$$\theta_1(x; t) = \sum_{n=-\infty}^{\infty} (-1)^n t^{\frac{n(n+1)}{2}} x^{n+\frac{1}{2}} = (t, t)_{\infty} (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) (tx; t)_{\infty} (t/x; t)_{\infty}.$$

Since, as was mentioned above, $\theta_3(x; t) = (t; t)_{\infty} (-\sqrt{t}x; t)_{\infty} (-\sqrt{t}/x; t)_{\infty}$, we have

$$\theta_1(x; t) = -\frac{1}{\sqrt{x}} \theta_3\left(-\frac{x}{\sqrt{t}}; t\right).$$

Corollary 2.8. *The dynamical correlation functions of the periodic Schur process in the analytic setting can be written as*

$$\begin{aligned} & \rho_n(\tau_1, x_1; \dots; \tau_n, x_n) \\ &= \frac{(t; t)_{\infty}^{3n}}{(2\pi i)^{2n}} \oint \dots \oint \frac{\prod_{1 \leq i < j \leq n} \theta_1(\zeta_i/\zeta_j; t) \theta_1(\eta_i/\eta_j; t)}{\prod_{i,j=1}^n \theta_1(\zeta_i \eta_j; t)} \prod_{i=1}^n \frac{F(\tau_i, \zeta_i)}{F(\tau_i, \eta_i^{-1})} \frac{d\zeta_i d\eta_i}{(\zeta_i \eta_i)^{x_i+1}}, \end{aligned}$$

where the integration variables ζ_i and η_i range over circles $|\zeta_i| = \alpha_i$, $|\eta_i| = \beta_i$ such that

$$\alpha_1 > \frac{1}{\beta_1} > \alpha_2 > \frac{1}{\beta_2} > \cdots > \alpha_n > \frac{1}{\beta_n}$$

and all the radii α_i and β_i are close enough to 1.

Proof. We start with the interpretation of Theorem 2.2 given in Remark 2.4 and note that

$$-\frac{\theta_3(z\zeta\eta;t)}{\theta_3(z;t)\theta_3(-\zeta\eta t^{-\frac{1}{2}};t)} = \frac{\theta_1(-z\sqrt{t}\zeta\eta;t)}{\theta_1(-z\sqrt{t};t)\theta_1(\zeta\eta;t)} = \frac{\theta_1(\widehat{z}\zeta\eta;t)}{\theta_1(\widehat{z};t)\theta_1(\zeta\eta;t)},$$

where we used the notation $\widehat{z} = -z\sqrt{t}$. The following Cauchy-type determinantal formula is due to Frobenius, see [R, Lemma 4.3], [F]:

$$\det \left[\frac{\theta_1(\widehat{z}\zeta_i\eta_j;t)}{\theta_1(\widehat{z};t)\theta_1(\zeta_i\eta_j;t)} \right]_{i,j=1}^n = \frac{\theta_1(\widehat{z}\prod_{i=1}^n \zeta_i\eta_i;t)}{\theta_1(\widehat{z};t)} \frac{\prod_{1 \leq i < j \leq n} \theta_1(\zeta_i/\zeta_j;t)\theta_1(\eta_i/\eta_j;t)}{\prod_{i,j=1}^n \theta_1(\zeta_i\eta_j;t)}.$$

Since

$$\frac{\theta_1(\widehat{z}\prod_{i=1}^n \zeta_i\eta_i;t)}{\theta_1(\widehat{z};t)} = \frac{\theta_3(z\prod_{i=1}^n \zeta_i\eta_i;t)}{\theta_3(z;t)\sqrt{\zeta_1\eta_1} \cdots \zeta_n\eta_n},$$

and the constant term in z of $\theta_3(z\prod_{i=1}^n \zeta_i\eta_i;t)$ is equal to 1, the application of Proposition 2.1 concludes the proof. \square

Note that Corollary 2.8 implies the generating series of the density function of the uniform measure on partitions is, up to a constant, the inverse of the first Jacobi theta-function:

$$\sum_{x \in \mathbb{Z}'} \rho_1(x) \xi^x = (t;t)_\infty / \theta_1(\xi;t).$$

Proof of Theorem 2.2. We will provide a proof for the algebraic variant of the theorem when both sides are considered as formal series, see Remark 2.3. The numeric equality in the analytic setting is a mere corollary: Under the convergence conditions stated in Remark 2.3 the formal series in both sides are absolutely convergent, and hence the fact that they coincide termwise implies that their sums are equal.

Our proof is based on the following well known statement:

Let \mathfrak{X} be a finite set; let L be a $|\mathfrak{X}| \times |\mathfrak{X}|$ matrix with rows and columns marked by the points of \mathfrak{X} with matrix elements from an algebra \mathcal{A} , and assume that $\det(1+L)$ is an invertible element of \mathcal{A} . Consider an \mathcal{A} -valued measure on the set $2^{\mathfrak{X}}$ of all subsets of \mathfrak{X} given by

$$\text{Prob}(X) = \frac{\det L_X}{\det(1+L)},$$

where L_X is the symmetric submatrix of L corresponding to X :

$$L_X = \|L(x_i, x_j)\|_{x_i, x_j \in X}.$$

Then the correlation functions of this measure are also given by minors of a matrix: For any $Y \subset \mathfrak{X}$

$$\rho(Y) = \text{Prob}\{X \in 2^{\mathfrak{X}} \mid Y \subset X\} = \det K_Y,$$

where $K = L(1 + L)^{-1}$.

Proofs of this statement can be found in [Macc, DVJ, BR]. Measures of the form above are often called *L-ensembles*.

Interesting examples of *L-ensembles* usually involve an infinite set \mathfrak{X} , and thus the above linear algebraic statement needs to be adjusted to the specific situation at hand. Our case is of the same nature.

Let us take $\mathfrak{X} = \mathbb{Z}' \sqcup \cdots \sqcup \mathbb{Z}'$ (total of $N + 1$ copies) and consider the matrix L which has the following block form corresponding to this splitting of \mathfrak{X} :

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & Q \\ -L[1] & 0 & 0 & \cdots & 0 & 0 \\ 0 & -L[2] & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -L[N] & 0 \end{bmatrix}.$$

The $\mathbb{Z}' \times \mathbb{Z}'$ matrices L_1, \dots, L_N are Toeplitz and their matrix elements are given by

$$(L[k])_{xy} = \text{coefficient of } \zeta^{x-y} \text{ in } \exp \sum_{n \geq 1} (a_n[k] \zeta^{-n} + b_n[k] \zeta^n),$$

while the matrix Q is diagonal: $Q_{xy} = z t^x \delta_{xy}$.

As the algebra \mathcal{A} we choose the algebra of formal series in $\Lambda^{\otimes(2N)}[t^{\pm 1}]$ which have at most a finite order pole at $t = 0$ (in other words, the degrees of t entering any element of \mathcal{A} must be bounded from below). We will also be using the subalgebra \mathcal{A}_{hol} of \mathcal{A} which consists of series holomorphic at $t = 0$ (i.e., the series which do not contain negative powers of t). The algebra \mathcal{A}_{hol} has a natural \mathbb{Z}_+ -filtration induced by the degrees of symmetric functions and polynomials in t . We will denote its filtered components by $\mathcal{A}_{hol}(d)$, $d \geq 0$. That is, $\mathcal{A}_{hol}(d)$ consists of series whose terms are of degree at least d . The algebra \mathcal{A} also has a natural topology: Two series are close if their difference is in $\mathcal{A}_{hol}(d)$ for d large enough. With respect to this topology the algebra \mathcal{A} is complete.

Recall also that the parameter z entering Q is considered numeric, which means that it does not contribute to the degree count.

The connection between the matrix L above and the periodic Schur process is explained by the following statement.

Lemma 2.9. *For any partitions λ and ν , an integer $l \geq \max\{\ell(\lambda), \ell(\nu)\}$, and any $k \in \{1, \dots, N\}$ we have*

$$\begin{aligned} & \det[(L[k])_{\nu_i - i, \lambda_j - j}]_{i,j=1}^l \\ &= H(a[k]; b[k]) \sum_{\mu \in \mathbb{Y}} s_{\lambda/\mu}(a[k]) s_{\nu/\mu}(b[k]) + \mathcal{O}(2l - \ell(\lambda) - \ell(\nu) + 2), \end{aligned}$$

where we use the notation $\mathcal{O}(d)$ for elements of $\mathcal{A}_{hol}(d)$.

Proof. To simplify the notation we will omit the index “ $[k]$ ” in the formulas below. By [Macd, Ex. I.5.26] we have

$$H(a; b) \sum_{\mu} s_{\lambda/\mu}(a) s_{\nu/\mu}(b) = \sum_{\rho} s_{\rho/\lambda}(b) s_{\rho/\nu}(a).$$

Let us split the sum in the right-hand side into two parts — over partitions ρ of length $\leq l$ and $> l$. The second part has terms of degree at least $2(l+1) - \ell(\lambda) - \ell(\nu)$ and thus can be ignored. The first part can be rewritten using the Jacobi-Trudi formula (see [Macd, §I (5.4)]) as:

$$\sum_{\rho_1 \geq \rho_2 \geq \dots \geq \rho_l \geq 0} \det[h_{\rho_i - i - \lambda_j + j}(b)]_{i,j=1}^l \det[h_{\rho_i - i - \nu_j + j}(a)]_{i,j=1}^l,$$

where h_n 's are the homogeneous symmetric functions; $h_n = 0$ for $n < 0$. By the Cauchy-Binet formula the last sum is readily seen to be equal to

$$\det \left[\sum_{k \geq -l} h_{k - \lambda_i + i}(b) h_{k - \nu_j + j}(a) \right]_{i,j=1}^l,$$

and the sum inside the determinant is equal to the needed matrix element of L because for $p, q \geq -l$ we have

$$\sum_{k \geq -l} h_{k-p}(b) h_{k-q}(a) = \sum_{k \in \mathbb{Z}} h_{k-p}(b) h_{k-q}(a) = \sum_{k \in \mathbb{Z}} h_{k+q-p}(b) h_k(a)$$

and

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} h_{k+n}(b) h_k(a) \zeta^n &= \sum_{k_1 \in \mathbb{Z}} h_{k_1}(b) \zeta^{k_1} \sum_{k_2 \in \mathbb{Z}} h_{k_2}(a) \zeta^{-k_2} \\ &= \exp \sum_{n \geq 1} (p_n(a) \zeta^{-n} + p_n(b) \zeta^n). \quad \square \end{aligned}$$

It is convenient to introduce separate notations for the two values of the kernel entering the statement of Theorem 2.2: Define $K_+(\sigma, x; \tau, y)$ and $K_-(\sigma, x; \tau, y)$ through the generating functions

$$\begin{aligned} \sum_{x, y \in \mathbb{Z}'} K_+(\sigma, x; \tau, y) \zeta^x \eta^y &= \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + (z t^m)^{-1}}, \\ \sum_{x, y \in \mathbb{Z}'} K_-(\sigma, x; \tau, y) \zeta^x \eta^y &= -\frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + z t^m}. \end{aligned}$$

We will denote by $K_{\pm}[\sigma, \tau]$ the $\mathbb{Z}' \times \mathbb{Z}'$ matrices with matrix elements

$$(K_{\pm}[\sigma, \tau])_{xy} = K_{\pm}(\sigma, x; \tau, y).$$

Here σ and τ are allowed to take values between 0 and N .

Lemma 2.10. (i) For any $\sigma \in \{0, \dots, N\}$ we have $K_+[\sigma, \sigma] = \mathbf{1} + K_-[\sigma, \sigma]$.

(ii) For any $\sigma, \tau \in \{0, \dots, N-1\}$ we have $L[\sigma+1]K_{\pm}[\sigma, \tau] = K_{\pm}[\sigma+1, \tau]$.

(iii) For any $\tau \in \{0, \dots, N\}$ we have $QK_-[N, \tau] = -K_+[0, \tau]$.

Proof. All these statements are proved by simple algebraic manipulations.

For (i) we have

$$\sum_{x,y \in \mathbb{Z}'} (K_+(\sigma, x; \sigma, y) - K_-(\sigma, x; \sigma, y)) \zeta^x \eta^y = \frac{F(\sigma, \zeta)}{F(\sigma, \eta^{-1})} \sum_{m \in \mathbb{Z}'} (\zeta \eta)^m = \sum_{m \in \mathbb{Z}'} (\zeta \eta)^m$$

whence $K_+[\sigma, \sigma] - K_-[\sigma, \sigma] = \mathbf{1}$.

The formula (ii) follows from the definitions of $L[\sigma]$, $K_\pm[\sigma, \tau]$, and the fact that

$$\left(\exp \sum_{n \geq 1} (a_n[\sigma + 1] \zeta^{-n} + b_n[\sigma + 1] \zeta^n) \right) F(\sigma, \zeta) = F(\sigma + 1, \zeta).$$

Finally, for (iii) we have

$$\begin{aligned} \sum_{x,y \in \mathbb{Z}'} (QK_-[N, \tau])_{xy} \zeta^x \eta^y &= -\frac{zF(N, t\zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(t\zeta \eta)^m}{1 + zt^m} \\ &= -\frac{F(0, \zeta)}{F(\tau, \eta^{-1})} \sum_{m \in \mathbb{Z}'} \frac{(\zeta \eta)^m}{1 + (zt^m)^{-1}} = -\sum_{x,y \in \mathbb{Z}'} (K_+[0, \tau])_{xy} \zeta^x \eta^y. \quad \square \end{aligned}$$

The relations of Lemma 2.10 immediately imply that if we introduce a $\mathfrak{X} \times \mathfrak{X}$ matrix K which has the block form

$$K = \begin{bmatrix} K_+[0, 0] & K_+[0, 1] & K_+[0, 2] & \dots & K_+[0, N] \\ K_-[1, 0] & K_+[1, 1] & K_+[1, 2] & \dots & K_+[1, N] \\ K_-[2, 0] & K_-[2, 1] & K_+[2, 2] & \dots & K_+[2, N] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_-[N, 0] & K_-[N, 1] & K_-[N, 2] & \dots & K_+[N, N] \end{bmatrix}$$

(the (σ, τ) -block is equal to $K_+[\sigma, \tau]$ if $\sigma \leq \tau$, and to $K_-[\sigma, \tau]$ otherwise), then we have the matrix relation $(\mathbf{1} + L)K = L$. In order to extract the probabilistic meaning of this relation we need to introduce certain finite point approximations of the shift-mixed periodic Schur process.

For any $m = 1, 2, \dots$ denote by $\mathbb{Z}'(m)$ the subset of \mathbb{Z}' consisting of $2m$ half-integers situated symmetrically around 0:

$$\mathbb{Z}'(m) = \left\{ -m + \frac{1}{2}, -m + \frac{3}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, \dots, m - \frac{3}{2}, m - \frac{1}{2} \right\}.$$

Denote also by $L^{(m)}[\sigma]$, $K_\pm^{(m)}[\sigma, \tau]$, and $Q^{(m)}$ the restrictions of the $\mathbb{Z}' \times \mathbb{Z}'$ matrices $L[\sigma]$, $K_\pm[\sigma, \tau]$, and Q to $\mathbb{Z}'(m) \times \mathbb{Z}'(m)$, and denote by $L^{(m)}$ and $K^{(m)}$ the block matrices built from $L^{(m)}[\sigma]$, $K_\pm^{(m)}[\sigma, \tau]$, $Q^{(m)}$ in the same way as L and K are built from $L[\sigma]$, $K_\pm[\sigma, \tau]$, Q . Set

$$\mathfrak{X}^{(m)} = \mathbb{Z}'(m) \sqcup \dots \sqcup \mathbb{Z}'(m) \quad (N+1 \text{ copies}).$$

Note that

$$L^{(m)} = L|_{\mathfrak{X}^{(m)} \times \mathfrak{X}^{(m)}}, \quad K^{(m)} = K|_{\mathfrak{X}^{(m)} \times \mathfrak{X}^{(m)}}.$$

Lemma 2.11. (i) $\det(1 + L^{(m)}) = z^m t^{-\frac{m^2}{2}} (1 + \mathcal{O}(1))$ where $\mathcal{O}(1)$ stands for an element of $\mathcal{A}_{hol}(1)$.

(ii) All matrix elements of $(1 + L^{(m)})^{-1}$ belong to \mathcal{A}_{hol} . Consequently, for any $X \subset \mathfrak{X}^{(m)}$ we have $\det L_X^{(m)} / \det(1 + L^{(m)}) \in \mathcal{A}_{hol}$.

Proof. (i) Since $\det(1 + L^{(m)}) = \sum_{X \subset \mathfrak{X}^{(m)}} \det L_X^{(m)}$, we need to see which minors $\det L_X^{(m)}$ yield terms of the lowest possible degree. Since

$$(L[k])_{xy} \in \mathcal{A}_{hol}(|x - y|) \quad \text{and} \quad (L[k])_{xx} = 1 + \mathcal{O}(1), \quad k = 1, \dots, N,$$

it is immediate that the only lowest degree term comes from $\det L_X^{(m)}$ with

$$X = \left\{-m + \frac{1}{2}, \dots, -\frac{1}{2}\right\} \sqcup \dots \sqcup \left\{-m + \frac{1}{2}, \dots, -\frac{1}{2}\right\}$$

($N + 1$ copies of the same set of negative elements in $\mathbb{Z}'(m)$), and it is equal to $z^m t^{-\frac{m^2}{2}}$ (recall that z does not contribute to the degree count).

(ii) Matrix elements of $(1 + L^{(m)})^{-1}$ are ratios of the linear combinations of minors of $L^{(m)}$ and $\det(1 + L^{(m)})$. Since all minors of $L^{(m)}$ lie in $t^{-\frac{m^2}{2}} \cdot \mathcal{A}_{hol}$, by (i) we see that the matrix elements belong to \mathcal{A}_{hol} .

As for the ratios $\det L_X / \det(1 + L^{(m)})$, observe that they are linear combinations of ratios of the form $\det(1 + L^{(m)})_X / \det(1 + L^{(m)})$ which coincide, up to a sign, with minors of the inverse matrix $(1 + L^{(m)})^{-1}$. \square

We are now in a position to prove that $L^{(m)}$ -ensembles on $\mathfrak{X}^{(m)}$ approximate the shift-mixed periodic Schur process on \mathfrak{X} as m becomes large.

Proposition 2.12. *The values of the correlation functions of the $L^{(m)}$ -ensembles restricted to the last N copies of $\mathbb{Z}'(m)$ in $\mathfrak{X}^{(m)} = \mathbb{Z}'(m) \sqcup \dots \sqcup \mathbb{Z}'(m)$ converge, as $m \rightarrow \infty$, to those of the dynamical correlation functions of the shift-mixed periodic Schur process in the topology of \mathcal{A} .*

Proof. In order to prove this statement we will construct an injective map that associates to any point configuration $X \subset \mathfrak{X}^{(m)}$ of nonzero weight in the $L^{(m)}$ -ensemble a sequence of partitions $(\lambda^{(1)}, \dots, \lambda^{(N)})$ and an integer S so that:

- The intersection of the point configuration

$$\{S + \lambda_i^{(1)} - i + \frac{1}{2}\}_{i \geq 1} \sqcup \dots \sqcup \{S + \lambda_i^{(N)} - i + \frac{1}{2}\}_{i \geq 1}$$

with $\mathbb{Z}'(m) \sqcup \dots \sqcup \mathbb{Z}'(m)$ coincides with the restriction of X to the last N copies of $\mathbb{Z}'(m)$ in $\mathfrak{X}^{(m)}$.

- The weight of X in the $L^{(m)}$ -ensemble and the weight of (λ, S) in the shift-mixed periodic Schur process are obtained from each other by multiplication by a constant of the form $C_m = 1 + \mathcal{O}(1)$ and by addition of an element of a high enough degree:

$$\frac{\det L_X^{(m)}}{\det(1 + L_X^{(m)})} = C_m \cdot z^S t^{\frac{S^2}{2}} \sum_{\mu} W(\lambda, \mu) + \mathcal{O}(d(m)),$$

where $d(m)$ does not depend on X and $d(m) \rightarrow \infty$ as $m \rightarrow \infty$.¹⁰

¹⁰We should have used $C_m / (\theta_3(z; t) Z(N, t, a, b))$ instead of C_m thus taking into account the partition function. However, this is equivalent because $\theta_3(z; t) Z(N, t, a, b)$ and $1 / (\theta_3(z; t) Z(N, t, a, b))$ are both of the form $1 + \mathcal{O}(1)$.

• The lowest degree of the weights of pairs (λ, S) not covered by this map goes to infinity as $m \rightarrow \infty$.

Since all the weights add up to 1 in both the $L^{(m)}$ -ensemble and the shift-mixed periodic Schur process, the existence of such a map implies that the degree of $C_m - 1$ goes to infinity as $m \rightarrow \infty$, and the needed convergence of the correlation functions readily follows.

Observe that if a set $X \subset \mathfrak{X}^{(m)}$ has a nonzero weight in the $L^{(m)}$ -ensemble (i.e., $\det L_X^{(m)} \neq 0$) then its intersections with all $N + 1$ copies of $\mathbb{Z}'(m)$ in $\mathfrak{X}^{(m)}$ must have the same cardinality because of the specific block structure of $L^{(m)}$, and its intersections with the first and the last copies of $\mathbb{Z}'(m)$ must coincide because the matrix Q is diagonal. Thus, without loss of generality we may assume that

$$X = \{x_1^{(0)}, \dots, x_l^{(0)}\} \sqcup \{x_1^{(1)}, \dots, x_l^{(1)}\} \sqcup \{x_1^{(2)}, \dots, x_l^{(2)}\} \sqcup \dots \sqcup \{x_1^{(N)}, \dots, x_l^{(N)}\}$$

for some $l \leq 2m$, where $x_i^{(0)} = x_i^{(N)}$ for all $1 \leq i \leq l$. Our notation for X means that the first group of points lies in the first copy of $\mathbb{Z}'(m)$, the second one lies in the second copy of $\mathbb{Z}'(m)$ and so on.

The element $(\lambda^{(1)}, \dots, \lambda^{(N)}; S)$ of the shift-mixed process corresponding to such X is defined as follows: $S = l - m$ and

$$\{S + \lambda_i^{(\tau)} - i + \frac{1}{2}\}_{i \geq 1} = \{x_1^{(\tau)}, \dots, x_l^{(\tau)}\} \cup \{-m - \frac{1}{2}, -m - \frac{3}{2}, -m - \frac{5}{2}, \dots\}$$

for any $\tau = 1, \dots, N$. It is readily seen that this formula correctly defines the partitions $\lambda^{(\tau)}$. Note that $\ell(\lambda^{(\tau)}) \leq l$ for all τ .

The pair $(\lambda^{(1)}, \dots, \lambda^{(N)}; S)$ is not covered by this map if and only if for some τ the point configuration $\{S + \lambda_i^{(\tau)} - i + \frac{1}{2}\}_{i \geq 1}$ either does not contain the set $\mathbb{Z}'_{<-m}$ or has a nonzero intersection with $\mathbb{Z}'_{>m}$. In the latter case we must have $\lambda_1^{(\tau)} + S \geq m + 1$ and in the former case we must have $(\lambda^{(\tau)})'_1 - S \geq m + 1$. This means that either $|S|$ or $|\lambda^{(\tau)}|$ is $\geq \frac{m}{2}$.

The definition of the shift-mixed process implies that the weight of (λ, S) has degree at least

$$\frac{S^2}{2} + ||\lambda^{(1)}| - |\lambda^{(2)}|| + \dots + ||\lambda^{(N-1)}| - |\lambda^{(N)}|| + |\lambda^{(N)}| \leq \frac{S^2}{2} + \max_{1 \leq \tau \leq N} |\lambda^{(\tau)}|.$$

Therefore, the pairs (λ, S) not in the image of our map have weights of degrees uniformly going to infinity as $m \rightarrow \infty$.

It remains to compare the weights of a point configuration $X \subset \mathfrak{X}^{(m)}$ in the $L^{(m)}$ -ensemble and of its image in the shift-mixed process.

Using Lemma 2.11(i) we obtain that the weight $\det L_X^{(m)} / \det(1 + L^{(m)})$ of X equals

$$\text{const} \cdot z^{-m} t^{\frac{m^2}{2}} \det[(L[1])_{x_i^{(1)}, x_j^{(0)}}]_{i,j=1}^l \cdots \det[(L[N])_{x_i^{(N)}, x_j^{(N-1)}}]_{i,j=1}^l \prod_{i=1}^l z t^{x_i^{(N)}}.$$

with $\text{const} = 1 + \mathcal{O}(1)$. Collecting the powers of z and t yields

$$\begin{aligned} & z^{l-m} t^{\frac{m^2}{2} + (S + \frac{1}{2})l + \sum_{i=1}^l (\lambda_i^{(N)} - i)} \\ &= z^{l-m} t^{\frac{m^2}{2} + (l-m+\frac{1}{2})l - \frac{l(l+1)}{2} + \sum_{i=1}^l \lambda_i^{(N)}} = z^S t^{\frac{S^2}{2} + |\lambda^{(N)}|} \end{aligned}$$

where we used the definition of $S = l - m$ and the fact that $\ell(\lambda^{(N)}) \leq l$.¹¹ Evaluating the minors of $L[\tau]$ by Lemma 2.9 (which is applicable because $\ell(\lambda^{(\tau)}) \leq l$) we obtain

$$\begin{aligned} & \det[(L[1])_{x_i^{(1)}, x_j^{(0)}}]_{i,j=1}^l \cdots \det[(L[N])_{x_i^{(N)}, x_j^{(N-1)}}]_{i,j=1}^l = \text{const} \\ & \times \left(\sum_{\mu^{(1)}} s_{\lambda^{(0)}/\mu^{(1)}}(a[1]) s_{\lambda^{(1)}/\mu^{(1)}}(b[1]) + \mathcal{O}(2l - \ell(\lambda^{(0)}) - \ell(\lambda^{(1)}) + 2) \right) \\ & \times \left(\sum_{\mu^{(2)}} s_{\lambda^{(1)}/\mu^{(2)}}(a[2]) s_{\lambda^{(2)}/\mu^{(2)}}(b[2]) + \mathcal{O}(2l - \ell(\lambda^{(1)}) - \ell(\lambda^{(2)}) + 2) \right) \cdots \\ & \times \left(\sum_{\mu^{(N)}} s_{\lambda^{(N-1)}/\mu^{(N)}}(a[N]) s_{\lambda^{(N)}/\mu^{(N)}}(b[N]) + \mathcal{O}(2l - \ell(\lambda^{(N-1)}) - \ell(\lambda^{(N)}) + 2) \right) \end{aligned}$$

with $\text{const} = 1 + \mathcal{O}(1)$. To conclude the proof we need to show that all the remainders $\mathcal{O}(\cdot)$ in the expression above can be removed by adding the correction of degree uniformly going to infinity as $m \rightarrow \infty$.

Observe that the degree of the factor $t^{\frac{S^2}{2} + |\lambda^{(N)}|}$ is bounded if and only if $|S| = |l - m|$ and $|\lambda^{(N)}|$ are bounded. Then in the last factor in the product above the term $\mathcal{O}(2l - \ell(\lambda^{(N-1)}) - \ell(\lambda^{(N)}) + 2)$ has degree going to infinity with m (because $l - \ell(\lambda^{(N)}) \geq m - S - |\lambda^{(N)}|$), so if we want the weight of X or the weight of (λ, S) to be of bounded degree, the degree of the sum over $\mu^{(N)}$ has to be bounded. Since

$$\deg \sum_{\mu} s_{\lambda/\mu}(a) s_{\nu/\mu}(b) \geq ||\lambda| - |\nu||,$$

this means that $|\lambda^{(N-1)}|$ has to be bounded. Repeating the argument with the second to last factor we conclude that $|\lambda^{(N-2)}|$ is bounded, and so on.

The final conclusion is that for the corresponding X and (λ, S) such that the minimum of the degrees of their weights is bounded, we must have $|l - m|$ and $|\lambda^{(\tau)}|$, $\tau = 1, \dots, N$, bounded. But then all $\mathcal{O}(\cdot)$'s in the product above have degrees uniformly going to infinity as $m \rightarrow \infty$, and thus the difference of the weights of X and (λ, S) has degree uniformly going to infinity as $m \rightarrow \infty$. The proof of Proposition 2.12 is complete. \square

Let us now conclude the proof of Theorem 2.2. From the determinantal formula for the correlation functions of general L -ensembles, see the beginning of the proof, we know that the correlation functions of the $L^{(m)}$ -ensembles are given by minors of the matrices $L^{(m)}(1 + L^{(m)})^{-1}$. The last step of the proof is to show that matrix elements of these matrices converge to those of the kernel K in the topology of \mathcal{A} .

The definitions of matrices $L[\sigma]$ and $K[\sigma, \tau]$ imply that

$$\deg(L[\sigma])_{xy} \geq |x - y|, \quad (K[\sigma, \tau])_{xy} \geq |x - y|, \quad x, y \in \mathbb{Z}',$$

for any σ and τ . Hence, from Lemma 2.10(ii) we obtain $(x, y \in \mathbb{Z}'(m))$

$$\begin{aligned} (L^{(m)}[\sigma + 1] K_{\pm}^{(m)}[\sigma, \tau])_{xy} &= (K_{\pm}^{(m)}[\sigma + 1, \tau])_{xy} \\ &+ \mathcal{O}(\min\{m + \tfrac{1}{2} - x, x - m - \tfrac{1}{2}\} + \min\{m + \tfrac{1}{2} - y, y - m - \tfrac{1}{2}\}). \end{aligned}$$

¹¹This is the computation mentioned in Remark 2.7 above.

The relations (i) and (iii) clearly remain unchanged when restricted to $\mathbb{Z}'(m)$:

$$K_+^{(m)}[\sigma, \sigma] = \mathbf{1} + K_-^{(m)}[\sigma, \sigma], \quad Q^{(m)}K_-^{(m)}[N, \tau] = -K_+^{(m)}[0, \tau].$$

Hence,

$$((1 + L^{(m)})K^{(m)})_{xy} = L_{xy}^{(m)} + \mathcal{O}(\min\{m + \frac{1}{2} - y, y - m - \frac{1}{2}\}).$$

Multiplying by $(\mathbf{1} + L^{(m)})^{-1}$ on the left and using Lemma 2.11(ii) we obtain

$$K_{xy}^{(m)} = ((1 + L^{(m)})^{-1}L^{(m)})_{xy} + \mathcal{O}(\min\{m + \frac{1}{2} - y, y - m - \frac{1}{2}\}).$$

Since $\min\{m + \frac{1}{2} - y, y - m - \frac{1}{2}\} \rightarrow \infty$ as $m \rightarrow \infty$ for any fixed y , the needed convergence follows. The proof of Theorem 2.2 is complete. \square

3. BULK SCALING LIMIT

As was mentioned in §1, the presence of the parameter t is crucial for defining the periodic Schur process; at $t = 1$ the partition function is, generally speaking, infinite. Thus, one might expect that if $t \rightarrow 1-$ then the random Young diagrams become large. Indeed, this is correct: In the analytic setting we will show that as $t \rightarrow 1-$, the density function of the scaled random point configuration $|\ln t| \cdot \mathcal{L}(\lambda^{(\tau)})$ tends to a nontrivial limit, and this limit is independent of τ . The main result of this section is the computation of the local limit of the correlation functions of the periodic Schur process and its shift-mixed modification near points of fixed global limit density.

Denote $r = \ln t^{-1}$. Throughout this section we assume that $r > 0$ (equivalently, $0 < t < 1$), and also that the convergence conditions of Remark 2.3 are satisfied. Namely, we assume that $|a_n[k]|, |b_n[l]| < \text{const} \cdot R^n$ for some $0 < R < 1$ and all $1 \leq k, l \leq N$, and we also assume that $|\arg(z)| < \pi$.

Recall the notation $A_k = \sum_{m=1}^N a_k[m]$, $B_k = \sum_{m=1}^N b_k[m]$, introduced in §1.

Theorem 3.1. (i) Assume that $A_k = \overline{B}_k$ for all $k = 1, 2, \dots$. Then, as $t \rightarrow 1$, the dynamical correlation functions of the shift-mixed periodic Schur process have a limit in the following sense: Choose $x_1(t), \dots, x_n(t) \in \mathbb{Z}'$ such that as $t \rightarrow 1$, $rx_k(t) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j = x_i(t) - x_j(t)$ are independent of t . Then for any $1 \leq \tau_1, \dots, \tau_n \leq N$

$$\lim_{t \rightarrow 1} \rho_n^{\text{shift}}(\tau_1, x_1(t); \dots, \tau_n, x_n(t)) = \det[\mathcal{K}_{\tau_i, \tau_j}^{(z, \gamma)}(x_i - x_j)]_{i, j=1}^n,$$

where the correlation kernel has the following form

$$\mathcal{K}_{\sigma, \tau}^{(z, \gamma)}(d) = \begin{cases} \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\exp(-\sum_{n \geq 1} \sum_{k=\sigma+1}^{\tau} (a_n[k]\zeta^{-n} + b_n[k]\zeta^n))}{1 + z^{-1} \exp(\gamma - \sum_{n \geq 1} (A_n \zeta^{-n} + B_n \zeta^n))} \frac{d\zeta}{\zeta^{d+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\exp(\sum_{n \geq 1} \sum_{k=\tau+1}^{\sigma} (a_n[k]\zeta^{-n} + b_n[k]\zeta^n))}{1 + z \exp(-\gamma + \sum_{n \geq 1} (A_n \zeta^{-n} + B_n \zeta^n))} \frac{d\zeta}{\zeta^{d+1}}, & \sigma > \tau. \end{cases}$$

(ii) Under the same assumptions the dynamical correlation functions of the periodic Schur process converge to the limiting expression above evaluated at $z = 1$:

$$\lim_{t \rightarrow 1} \rho_n(\tau_1, x_1(t); \dots, \tau_n, x_n(t)) = \det[\mathcal{K}_{\tau_i, \tau_j}^{(1, \gamma)}(x_i - x_j)]_{i, j=1}^n.$$

Comments. 1. The limit correlation functions as functions on $\mathbb{Z} \sqcup \dots \sqcup \mathbb{Z}$ (N copies) are invariant with respect to the simultaneous shifts of all variables.

2. It will be clear from the proof that a slightly more general statement is true: For several groups of variables $\{x_k^{(m)}(t)\}_{k=1}^{n_m}$, $m = 1, \dots, M$, such that $rx_k^{(m)} \rightarrow \gamma_m$, the pairwise distances inside every group are independent of t , and the distances between different groups tend to infinity:

$$x_i^{(m)}(t) - x_j^{(m)}(t) = x_i^{(m)} - x_j^{(m)}, \quad \min_{i,j} |x_i^{(m_1)}(t) - x_j^{(m_2)}(t)| \rightarrow \infty \text{ for } m_1 \neq m_2,$$

the limit of the dynamical correlation functions of the shift-mixed process is the product of determinants:

$$\lim_{t \rightarrow 1} \rho_n^{\text{shift}}(\{\tau_i^{(m)}, x_i^{(m)}(t)\}_{1 \leq i \leq n_m, 1 \leq m \leq M}) = \prod_{m=1}^M \det \left[\mathcal{K}_{\tau_i^{(m)}, \tau_j^{(m)}}^{(z, \gamma_m)}(x_i^{(m)} - x_j^{(m)}) \right]_{i,j=1}^{n_m},$$

and the same is true for the correlation functions of the periodic Schur process with $z = 1$ in the right-hand side of the formula above. Roughly speaking, this means that particles become independent as the distance between them grows.

3. The global limit density function mentioned in the beginning of the section is equal to

$$\rho^{(z)}(\gamma) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{1}{1 + z^{-1}e^{\gamma - \sum_{n \geq 1} (A_n \zeta^{-n} + B_n \zeta^n)}} \frac{d\zeta}{\zeta}$$

for the shift-mixed process, and to $\rho^{(1)}(\gamma)$ for the unmixed process. This formula has the following corollary: If one assumes the existence of the limit shape, as $t \rightarrow 1$, of the random Young diagrams $\lambda^{(\tau)}$ distributed according to the periodic Schur process, then this limit shape can be easily guessed. Denote by i and j the row and column coordinates on the diagrams and introduce new coordinates $u = r(j - i)$ and $v = r(i + j)$. Then the equation for the boundary of the hypothetical limit shape has the form

$$v(u) = u + 2 \oint_{|\zeta|=1} \ln \left(1 + e^{-u + \sum_{n \geq 1} (A_n \zeta^{-n} + B_n \zeta^n)} \right) \frac{d\zeta}{\zeta}.$$

This formula follows from the relation $\rho(u) = \frac{1}{2}(1 - v'(u))$, see [BOO, Remark 1.7] for an explanation.

4. In the case of the uniform measure on partitions, when the specializations $a[k], b[k]$ are trivial, Theorem 3.1 (or rather its extended version from Comment 1 above) coincides with Theorem 7 of [O1]. In this case the limit correlation kernel degenerates:

$$\mathcal{K}_{\sigma, \tau}^{(z, \gamma)}(d) = \begin{cases} \delta_{0d} \cdot (1 + z^{-1}e^{\gamma})^{-1}, & \sigma \leq \tau, \\ -\delta_{0d} \cdot (1 + ze^{-\gamma})^{-1}, & \sigma > \tau. \end{cases}$$

For the uniform measure the limit shape does exist, see [V], and the formula for $v(u)$ from the previous comment produces the correct answer: $v(u) = u + 2 \ln(1 + e^{-u})$ or $e^{\frac{v-u}{2}} + e^{\frac{u+v}{2}} = 1$.

Proof of Theorem 3.1. (i) We start with the integral representation of the correlation kernel for the shift-mixed process from Remark 2.4. Replacing the integration variable η by $\xi = \zeta\eta$ we obtain

$$K(\sigma, x; \tau, y) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)^2} \oint_{|\zeta|=1} \oint_{|\xi|=1 \pm \epsilon} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})} \frac{\theta_3(z\xi; t)}{\theta_3(-\xi t^{-\frac{1}{2}}; t)} \frac{d\zeta d\xi}{\zeta^{x-y+1} \xi^{y+\frac{1}{2}}},$$

where $\epsilon > 0$ is much smaller than r , and we choose $|\xi| = 1 + \epsilon$ for $\sigma \leq \tau$ and $|\xi| = 1 - \epsilon$ for $\sigma > \tau$. The next step is to fix an arbitrary ζ on the unit circle and to evaluate the asymptotics of the integral over ξ .

Proposition 3.2. (i) Assume that $|\arg z| < \pi$. Then, as $t \rightarrow 1$, the function

$$f(\xi) = -\frac{\prod_{n \geq 1} (1 - t^n)^3 \theta_3(z\xi; t)}{\theta_3(z; t) \theta_3(-\xi t^{-\frac{1}{2}}; t)}$$

on the circle $|\xi| = 1 \pm \epsilon$ uniformly converges to 0 on the complement to any neighborhood of the point $\xi = 1 \pm \epsilon$. On the other hand, there exists $\delta > 0$ such that for $|\arg(\xi)| < \delta$

$$f(\xi) = \frac{2\pi i \xi^{-\frac{1}{2}}}{r} \frac{e^{\frac{\ln(z) \ln(\xi)}{r}}}{e^{\frac{\pi i \ln(\xi)}{r}} - e^{-\frac{\pi i \ln(\xi)}{r}}} \cdot (1 + f_0(\xi))$$

where $f_0(\xi)$ is an analytic function which, as $t \rightarrow 1$, uniformly converges to 0 while $r f'_0(\xi)$ remains uniformly bounded.¹² All the estimates are uniform in z varying in a compact set of the complex plane bounded away from the negative real semiaxis.

(ii) Assume that z lies on the unit circle $|z| = 1$. Then on the circle $|\xi| = 1 \pm \epsilon$, as $t \rightarrow 1$, we have the bound $|f(\xi)| \leq \text{const} \cdot \epsilon^{-1}$ which is uniform in both z and ξ .

Proof. (i) Denote $v = \frac{1}{2\pi i} \ln \xi$ and $w = \frac{1}{2\pi i} \ln z$. Applying the imaginary Jacobi transform, see e.g. [Er, 13.22(8)], we obtain

$$\begin{aligned} \theta_3(z; t) &= \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{2\pi^2 w^2}{r}} \theta_3\left(e^{\frac{4\pi^2 w}{r}}; e^{-\frac{4\pi^2}{r}}\right), \\ \theta_3(z\xi; t) &= \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{2\pi^2 (v+w)^2}{r}} \theta_3\left(e^{\frac{4\pi^2 (v+w)}{r}}; e^{-\frac{4\pi^2}{r}}\right), \\ \theta_3(-\xi t^{-\frac{1}{2}}; t) &= \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{2\pi^2 (v - \frac{ir}{4\pi} + \frac{1}{2})^2}{r}} \theta_3\left(-e^{\frac{4\pi^2 (v + \frac{1}{2})}{r}}; e^{-\frac{4\pi^2}{r}}\right). \end{aligned}$$

Hence,

$$f(\xi) = \frac{-\left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \prod_{n \geq 1} (1 - t^n)^3 e^{-\frac{2\pi^2}{r}((v+w)^2 - w^2 - (v - \frac{ir}{4\pi} + \frac{1}{2})^2)} \theta_3\left(e^{\frac{4\pi^2 (v+w)}{r}}; e^{-\frac{4\pi^2}{r}}\right)}{\theta_3\left(e^{\frac{4\pi^2 w}{r}}; e^{-\frac{4\pi^2}{r}}\right) \theta_3\left(-e^{\frac{4\pi^2 (v + \frac{1}{2})}{r}}; e^{-\frac{4\pi^2}{r}}\right)}.$$

We have

$$\begin{aligned} e^{-\frac{2\pi^2}{r}((v+w)^2 - w^2 - (v - \frac{ir}{4\pi} + \frac{1}{2})^2)} &= -ie^{\frac{2\pi^2}{r}(\frac{1}{4} + v(1-2w)) - \pi i v} \cdot e^{-\frac{r}{8}}, \\ \lim_{t \rightarrow 1} \frac{r^{\frac{3}{2}} e^{\frac{\pi^2}{2r}}}{(2\pi)^{\frac{3}{2}}} \prod_{n \geq 1} (1 - t^n)^3 &= 1. \end{aligned}$$

The last relation can be obtained, for example, from the imaginary Jacobi transform of $\theta_1(x; t)$ because $\theta_1(x; t) \sim (x^{\frac{1}{2}} - x^{-\frac{1}{2}})(t; t)_\infty^3$ as $x \rightarrow 1$. Thus,

$$-\left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \prod_{n \geq 1} (1 - t^n)^3 e^{-\frac{2\pi^2}{r}((v+w)^2 - w^2 - (v - \frac{ir}{4\pi} + \frac{1}{2})^2)} = \frac{2\pi i e^{\frac{2\pi^2 v(1-2w)}{r} - \pi i v} (1 + o(1))}{r}.$$

¹²Here we use the principal branch of the logarithm function for both $\ln(z)$ and $\ln(\xi)$.

Writing down the products for θ_3 's explicitly we obtain

$$\begin{aligned}\theta_3\left(e^{\frac{4\pi^2 w}{r}}; e^{-\frac{4\pi^2}{r}}\right) &= \Pi_0 \cdot \prod_{n \geq 0} \left(1 + e^{\frac{4\pi^2(w-n-\frac{1}{2})}{r}}\right) \left(1 + e^{\frac{4\pi^2(-w-n-\frac{1}{2})}{r}}\right), \\ \theta_3\left(e^{\frac{4\pi^2(v+w)}{r}}; e^{-\frac{4\pi^2}{r}}\right) &= \Pi_0 \cdot \left(1 + e^{\frac{4\pi^2(v+w-\frac{1}{2})}{r}}\right) \left(1 + e^{\frac{4\pi^2(-v-w-\frac{1}{2})}{r}}\right) \\ &\quad \times \prod_{n \geq 0} \left(1 + e^{\frac{4\pi^2(v+w-n-\frac{3}{2})}{r}}\right) \left(1 + e^{\frac{4\pi^2(-v-w-n-\frac{3}{2})}{r}}\right), \\ \theta_3\left(-e^{\frac{4\pi^2(v+\frac{1}{2})}{r}}; e^{-\frac{4\pi^2}{r}}\right) &= \Pi_0 \cdot \left(1 - e^{\frac{4\pi^2 v}{r}}\right) \\ &\quad \times \prod_{n \geq 0} \left(1 - e^{\frac{4\pi^2(v-n-1)}{r}}\right) \left(1 - e^{\frac{4\pi^2(-v-n-1)}{r}}\right),\end{aligned}$$

where $\Pi_0 = \prod_{n \geq 1} (1 - e^{-\frac{4\pi^2 n}{r}}) \rightarrow 1$ as $r \rightarrow 0$. Using the principal branch of the logarithm we may assume that

$$\Re v = \frac{1}{2\pi} \arg \xi \in [-\frac{1}{2}, \frac{1}{2}] \quad \text{and} \quad \Re w = \frac{1}{2\pi} \arg z \in (-\frac{1}{2}, \frac{1}{2})$$

(recall that $|\arg z| < \pi$ by assumption).

It is readily seen that under these conditions, as $r \rightarrow 0$, all three products over $n \geq 0$ in the formulas above uniformly tend to 1 while their derivatives with respect to v multiplied by r are uniformly bounded. The remaining factors are

$$f(\xi) \sim \frac{2\pi i e^{\frac{2\pi^2 v(1-2w)}{r}} - \pi i v \left(1 + e^{\frac{4\pi^2(v+w-\frac{1}{2})}{r}}\right) \left(1 + e^{\frac{4\pi^2(-v-w-\frac{1}{2})}{r}}\right)}{r \left(1 - e^{\frac{4\pi^2 v}{r}}\right)}.$$

Assume first that $\Re v$ is bounded away from 0 (that is, ξ is bounded away from $\xi = 1 \pm \epsilon$). Note that since $\epsilon \ll r$, we have $\Im v = \frac{1}{2\pi} \ln |1 \pm \epsilon| = o(r)$ and

$$\Re(v(1-2w)) = \Re v(1-2\Re w) + o(r).$$

If $\Re v < 0$ and $\Re(v+w) \geq -\frac{1}{2}$ then the absolute value of the asymptotic expression for $f(\xi)$ above is bounded by

$$\text{const} \cdot r^{-1} e^{2\pi^2 \Re v(1-2\Re w)r^{-1}}$$

which uniformly converges to 0 as $r \rightarrow 0$. If $\Re v < 0$ and $\Re(v+w) < -\frac{1}{2}$ then the bound is

$$\text{const} \cdot r^{-1} e^{2\pi^2(\Re v(1-2\Re w)-2\Re v-2\Re w-1)r^{-1}} = \text{const} \cdot r^{-1} e^{-2\pi^2(\Re v+1)(2\Re w+1)r^{-1}}$$

which also goes to 0 as $r \rightarrow 0$. If $\Re v > 0$ and $\Re(v+w) \leq \frac{1}{2}$ then $|f(\xi)|$ is bounded by

$$\text{const} \cdot r^{-1} e^{2\pi^2(\Re v(1-2\Re w)-2\Re v)r^{-1}} = \text{const} \cdot r^{-1} e^{-2\pi^2 \Re v(1+2\Re w)r^{-1}}$$

which goes to 0 as $r \rightarrow 0$. Finally, if $\Re v > 0$ and $\Re(v+w) > \frac{1}{2}$ then the bound has the form

$$\text{const} \cdot r^{-1} e^{2\pi^2(\Re v(1-2\Re w)+2(\Re v+\Re w-\frac{1}{2})-2\Re v)r^{-1}} = \text{const} \cdot r^{-1} e^{-2\pi^2(\Re v-1)(1-2\Re w)r^{-1}}$$

which is also small as $r \rightarrow 0$. This takes care of the first statement (about uniform convergence to 0) of Proposition 3.2(i).

Since $|\Re w| < \frac{1}{2}$, we can choose $\delta > 0$ such that $|\Re w \pm \frac{\delta}{2\pi}| < \frac{1}{2}$. Then for ξ with $|\arg(\xi)| = 2\pi|\Re v| < \delta$ we have $\Re(v+w-\frac{1}{2}) < 0$, and the two factors in the numerator of the approximation for $f(\xi)$ above uniformly converge to 1 as $r \rightarrow 0$ with their derivatives with respect to v multiplied by r are uniformly bounded. Thus,

$$f(\xi) \sim \frac{2\pi i e^{-\frac{4\pi^2 v w}{r} - \pi i v}}{r \left(e^{\frac{-2\pi^2 v}{r}} - e^{\frac{2\pi^2 v}{r}} \right)}$$

as desired. It is also immediately visible that all the estimates above are uniform in z varying in a compact set bounded away from the negative semiaxis. The proof of (i) is complete.

The proof of (ii) follows the arguments used above to prove the first part of (i). There are two differences in estimates: the range of $\Re w$ is now the whole segment $[-\frac{1}{2}, \frac{1}{2}]$ rather than a closed subset of the interval $(-\frac{1}{2}, \frac{1}{2})$, and since $\Re v$ is not bounded away from 0, we have an additional potentially small factor of the form $(1 - e^{\pm \frac{4\pi^2 v}{r}})$ in the denominator. We have

$$\arg(e^{\pm \frac{4\pi^2 v}{r}}) = \pm \frac{4\pi^2 \Im v}{r} = \pm \frac{2\pi}{r} \ln(1 \pm \epsilon) \sim \pm \frac{2\pi\epsilon}{r}$$

and hence $|1 - e^{\pm \frac{4\pi^2 v}{r}}| \geq \text{const} \frac{2\pi\epsilon}{r}$. Adding this estimate to those derived above yields the needed bound. \square

Our next step is to compute the asymptotics of the ratio $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$ from the integral representation of the correlation kernel given in the beginning of the proof of Theorem 3.1. Recall that the definition of the function $F(\tau, \zeta)$ was given just before Theorem 2.2. After simple manipulations, using the assumption that $B_k = \bar{A}_k$ for $k \geq 1$, we obtain

$$\begin{aligned} \frac{F(\sigma, \zeta)}{F(\tau, \zeta\xi^{-1})} &= \exp \sum_{n \geq 1} \frac{1}{1 - t^n} \left(\bar{A}_n \zeta^n (1 - \xi^{-n}) - A_n \zeta^{-n} (1 - \xi^n) \right) \\ &\times \exp \sum_{n \geq 1} \left(\zeta^{-n} \sum_{k=1}^{\sigma} a_n[k] - \zeta^n \sum_{k=\sigma+1}^N b_n[k] - (\zeta/\xi)^{-n} \sum_{k=1}^{\tau} a_n[k] + (\zeta/\xi)^n \sum_{k=\tau+1}^N b_n[k] \right). \end{aligned}$$

For any σ and τ this expression viewed as a function in (ζ, ξ) ranging over the circles $|\zeta| = 1$, $|\xi| = 1 \pm \epsilon$, remains uniformly bounded away from 0 and ∞ as $t \rightarrow 1$ if $0 < \epsilon \ll r$. Indeed, the boundedness of the second factor is obvious, while for the first factor we have

$$1 - t^n \geq n r t^n, \quad |\xi^{\pm n} - (\xi/|\xi|)^{\pm n}| = |(1 \pm \epsilon)^{\pm n} - 1| \leq n \epsilon (1 + \epsilon)^n$$

and hence

$$\begin{aligned} \sum_{n \geq 1} \left| \Re \frac{\bar{A}_n \zeta^n (1 - \xi^{-n}) - A_n \zeta^{-n} (1 - \xi^n)}{1 - t^n} \right| &= \sum_{n \geq 1} \left| \Re \frac{\bar{A}_n (\zeta/\xi)^n - A_n (\zeta/\xi)^{-n}}{1 - t^n} \right| \\ &\leq 2\epsilon r^{-1} \sum_{n \geq 1} t^{-n} (1 + \epsilon)^n |A_n| \end{aligned}$$

is bounded as $t \rightarrow 1$ because $\epsilon r^{-1} \rightarrow 0$, and $|A_n| = O(R^n)$ for a fixed $R < 1$ as $n \rightarrow \infty$. Similar arguments imply that the derivative $\frac{d}{d\xi} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})}$ multiplied by r is uniformly bounded.

Let us now look more carefully at the ratio in question when ξ is close to the real line. Assume that $|\arg \xi| \leq \alpha$ with $r \ll \alpha \ll r^{\frac{1}{2}}$. Then, using the estimates

$$\begin{aligned} |\xi^{\pm n} - 1 \mp n \ln \xi| &\leq (n |\ln \xi|)^2 \max\{1, |\xi|^n\} \leq \text{const} \cdot n^2 \alpha^2 (1 + \epsilon)^n \ll n^2 r (1 + \epsilon)^n, \\ \left| (1 - t^n)^{-1} - \frac{1}{nr} \right| &= \frac{|t^n - 1 - nr|}{(1 - t^n) nr} \leq \frac{n^2 r^2}{nr t^n nr} = t^{-n}, \end{aligned}$$

and the fact that $|A_n| = O(R^n)$ as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})} &= (1 + F_0(\xi)) \cdot \exp\left(r^{-1} \ln \xi \sum_{n \geq 1} (\bar{A}_n \zeta^n + A_n \zeta^{-n})\right) \\ &\times \exp \sum_{n \geq 1} \left(\zeta^{-n} \sum_{k=1}^{\sigma} a_n[k] - \zeta^{-n} \sum_{k=1}^{\tau} a_n[k] - \zeta^n \sum_{k=\sigma+1}^N b_n[k] + \zeta^n \sum_{k=\tau+1}^N b_n[k] \right), \end{aligned}$$

where $F_0(\xi)$ uniformly converges to 0 as $r \rightarrow 0$. Differentiating this relation with respect to ξ and using the boundedness of $\frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})}$ and $r \frac{d}{d\xi} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})}$ as well as the fact that

$$\left| \exp\left(r^{-1} \ln \xi \sum_{n \geq 1} (\bar{A}_n \zeta^n + A_n \zeta^{-n})\right) \right| = \exp\left(r^{-1} \ln(1 \pm \epsilon) \sum_{n \geq 1} (\bar{A}_n \zeta^n + A_n \zeta^{-n})\right) \rightarrow 0$$

as $r \rightarrow 0$, we see that $r F'_0(\xi)$ is uniformly bounded in both ξ ranging over the arch $|\arg \xi| \leq \alpha$, $|\xi| = 1 \pm \epsilon$, and ζ ranging over the unit circle.

We can now proceed to evaluating the asymptotics of the integral over ξ . Set

$$G_{\pm}(\zeta) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)} \oint_{|\xi|=1 \pm \epsilon} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})} \frac{\theta_3(z\xi; t)}{\theta_3(-\xi t^{-\frac{1}{2}}; t)} \frac{d\xi}{\xi^{y+\frac{1}{2}}}.$$

According to the hypothesis of Theorem 3.1, we assume that $y = \gamma r^{-1} (1 + o(1))$. Note that $|\xi^{y+\frac{1}{2}}| \rightarrow 1$ as $r \rightarrow 0$, because $\epsilon r \rightarrow 0$.

Let us split the circle $|\xi| = 1 \pm \epsilon$ into three parts: $|\arg \xi| \geq \delta$ with $\delta > 0$ taken from Proposition 3.2(i); $\alpha \leq |\arg \xi| < \delta$ with $r \ll \alpha \ll r^{\frac{1}{2}}$; and $|\arg \xi| < \alpha$.

The integral over the first part tends to zero (uniformly in ζ on the unit circle) because $f(\xi)$ of Proposition 3.1 converges to 0 and the factors $\frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})}$ and $\xi^{-y-\frac{1}{2}}$ remain bounded.

The integral over the second part, by virtue of the boundedness of the same factors and the asymptotics of $f(\xi)$ from Proposition 3.1(i), is bounded by (recall the notation $v = \frac{1}{2\pi i} \ln \xi$)

$$\frac{\text{const}}{r} \int_{v + \frac{i \ln(1 \pm \epsilon)}{2\pi} \in (-\delta, -\frac{\alpha}{2\pi}] \cup [\frac{\alpha}{2\pi}, \delta)} \left| \frac{e^{\frac{2\pi i \ln(z)v}{r}}}{e^{\frac{-2\pi^2 v}{r}} - e^{\frac{2\pi^2 v}{r}}} \right| dv.$$

Introducing a new variable $\hat{v} = 2\pi^2 r^{-1} v$, we can rewrite this expression as

$$\text{const} \int_{\hat{v} + \frac{i r \ln(1 \pm \epsilon)}{4\pi^2} \in (-\frac{2\pi^2 \delta}{r}, -\frac{\pi \alpha}{r}] \cup [\frac{\pi \alpha}{r}, \frac{2\pi^2 \delta}{r})} \left| \frac{e^{\frac{i \ln(z) \hat{v}}{\pi}}}{e^{-\hat{v}} - e^{\hat{v}}} \right| d\hat{v},$$

which tends to 0 as $r \rightarrow 0$ as long as $|\arg z| < \pi$, $\epsilon r \rightarrow 0$, and $\alpha/r \rightarrow \infty$.

The only nonzero contribution comes from the third part. Using the variable $\hat{v} = 2\pi^2 r^{-1} v = -i\pi r^{-1} \ln \xi$, the estimates for the ratio $\frac{F(\sigma, \zeta)}{F(\tau, \zeta^{\epsilon-1})}$ obtained above, and Proposition 3.1(i), we can write the integral over the third part $|\arg \xi| < \alpha$ in the form (note that $d\xi = \frac{ir\xi}{\pi} d\hat{v}$)

$$\begin{aligned} & \exp \sum_{n \geq 1} \left(\zeta^{-n} \sum_{k=1}^{\sigma} a_n[k] - \zeta^{-n} \sum_{k=1}^{\tau} a_n[k] - \zeta^n \sum_{k=\sigma+1}^N b_n[k] + \zeta^n \sum_{k=\tau+1}^N b_n[k] \right) \\ & \times \frac{1}{i\pi} \int_{\hat{v} + \frac{i r \ln(1 \pm \epsilon)}{4\pi^2} \in (-\frac{\pi \alpha}{r}, \frac{\pi \alpha}{r})} \frac{e^{\frac{i\hat{v}}{\pi} (\ln(z) - \gamma + \sum_{n \geq 1} (\bar{A}_n \zeta^n + A_n \zeta^{-n}))}}{e^{\hat{v}} - e^{-\hat{v}}} (1 + G_0(\hat{v})) d\hat{v}, \end{aligned}$$

where the function $G_0(\hat{v})$ uniformly converges to 0 as $t \rightarrow 1$, and the derivative $G'_0(\hat{v})$ remains uniformly bounded. The statement of Theorem 3.1(i) is a corollary of this formula, the above estimates, and the following lemma.

Lemma 3.3. *For any $a \in \mathbb{C}$ such that $|\Im a| < 1$ the following limit relation holds:*

$$\lim_{\varepsilon \rightarrow 0+} \int_{-\infty \pm i\varepsilon}^{+\infty \pm i\varepsilon} \frac{e^{iax}}{e^x - e^{-x}} (1 + g(\varepsilon, x)) dx = \frac{\mp i\pi}{1 + e^{\pm \pi a}},$$

where we assume that the function $g(\varepsilon, x)$ on $\mathbb{R} \pm i\varepsilon$ uniformly tends to 0 as $\varepsilon \rightarrow 0+$, while its derivative $\frac{d}{dx} g(\varepsilon, x)$ remains uniformly bounded. The convergence is uniform in a varying over any compact subset of \mathbb{C} with $\max |\Im a| < 1$.

Proof. The results with $x \in \mathbb{R} + i\varepsilon$ and $x \in \mathbb{R} - i\varepsilon$ are obtained from each other by the change of sign of the integration variable. Hence, we may assume that x varies over $\mathbb{R} - i\varepsilon$.

Let us handle the term with $g(\varepsilon, x)$ first. Split the integral into three parts:

$$\int_{-\infty - i\varepsilon}^{-x_0 - i\varepsilon} \frac{e^{iax} g(x, \varepsilon) dx}{e^x - e^{-x}} = \int_{-\infty - i\varepsilon}^{-x_0 - i\varepsilon} + \int_{-x_0 - i\varepsilon}^{x_0 - i\varepsilon} + \int_{x_0 - i\varepsilon}^{+\infty - i\varepsilon}$$

for some $x_0 = x_0(\varepsilon) \in \mathbb{R}$ that will be chosen later. Denote $M(\varepsilon) = \sup_{x \in \mathbb{R} - i\varepsilon} |g(x, \varepsilon)|$. The third integral can be estimated as follows:

$$\begin{aligned} \left| \int_{x_0 - i\varepsilon}^{+\infty - i\varepsilon} \frac{e^{iax} g(x, \varepsilon) dx}{e^x - e^{-x}} \right| &\leq M(\varepsilon) \int_{x_0 - i\varepsilon}^{+\infty - i\varepsilon} \frac{e^{\Re(ax)} dx}{e^{\Re x} - e^{-\Re x}} \\ &= e^{\varepsilon \Re a} M(\varepsilon) \int_{x_0}^{+\infty} \frac{e^{\Im(a)y} dy}{e^y - e^{-y}} \leq e^{\varepsilon \Re a} M(\varepsilon) \int_{x_0}^{+\infty} \frac{dy}{e^{(1-|\Im a|)y} - 1} \\ &= \frac{e^{\varepsilon \Re a} M(\varepsilon)}{1 - |\Im a|} \ln \left(\frac{e^{(1-|\Im a|)x_0}}{e^{(1-|\Im a|)x_0} - 1} \right). \end{aligned}$$

If $x_0(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then the last expression is bounded by $\text{const} \cdot M(\varepsilon) |\ln x_0(\varepsilon)|$, which tends to 0 if $x_0(\varepsilon) > e^{-\frac{1}{M(\varepsilon)}}$. A similar estimate holds for the first integral. As for the second integral, using the notation $M'(\varepsilon) = \sup_{(-x_0 - i\varepsilon, x_0 - i\varepsilon)} \left| \frac{d}{dx} (e^{iax} g(x, \varepsilon)) \right|$, we have

$$\begin{aligned} \left| \int_{-x_0 - i\varepsilon}^{x_0 - i\varepsilon} \frac{e^{iax} g(x, \varepsilon) dx}{e^x - e^{-x}} \right| &\leq M'(\varepsilon) \int_{-x_0}^{x_0} \frac{|y| dy}{|e^{y-i\varepsilon} - e^{-y+i\varepsilon}|} + \left| \int_{-x_0}^{x_0} \frac{g(-i\varepsilon, \varepsilon) dy}{e^{y-i\varepsilon} - e^{-y+i\varepsilon}} \right| \\ &\leq \text{const } M'(\varepsilon) x_0 + \frac{M(\varepsilon)}{2} \left| \ln(e^{x_0 - i\varepsilon} - 1) - \ln(e^{-x_0 + i\varepsilon} - 1) - \ln \frac{e^{x_0 - i\varepsilon} + 1}{e^{-x_0 + i\varepsilon} + 1} \right| \\ &\leq \text{const}(M'(\varepsilon) x_0 + M(\varepsilon)), \end{aligned}$$

which tends to 0 if $x(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, we proved that the term with $g(x, \varepsilon)$ tends to 0 as $\varepsilon \rightarrow 0$.

The remaining term gives (we use $\hat{y} = \cosh(y)/\sin \varepsilon$ below)

$$\begin{aligned} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{e^{iax} dx}{e^x - e^{-x}} &= e^{a\varepsilon} \int_{-\infty}^{\infty} \frac{e^{ia y} dy}{e^{y-i\varepsilon} - e^{-y+i\varepsilon}} \\ &= ie^{a\varepsilon} \sin \varepsilon \int_{-\infty}^{\infty} \frac{\cos(ay)(e^y + e^{-y}) dy}{(e^y - e^{-y})^2 + 4 \sin^2 \varepsilon} + ie^{a\varepsilon} \cos \varepsilon \int_{-\infty}^{\infty} \frac{\sin(ay)(e^y - e^{-y}) dy}{(e^y - e^{-y})^2 + 4 \sin^2 \varepsilon} \\ &\rightarrow \frac{i}{2} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{\cos(a \cdot \text{arccosh}(\hat{y} \sin \varepsilon)) d\hat{y}}{\hat{y}^2 + 1} + \int_{-\infty}^{\infty} \frac{\sin(ay) dy}{e^y - e^{-y}} \\ &= \frac{i\pi}{2} + \frac{i\pi}{2} \frac{e^{\pi a} - 1}{e^{\pi a} + 1} = \frac{i\pi}{1 + e^{-\pi a}}. \quad \square \end{aligned}$$

Lemma 3.3 implies that as $t \rightarrow 1$,

$$\begin{aligned} G_{\pm}(\zeta) &\rightarrow \pm \exp \sum_{n \geq 1} \left(\zeta^{-n} \sum_{k=1}^{\sigma} a_n[k] - \zeta^{-n} \sum_{k=1}^{\tau} a_n[k] - \zeta^n \sum_{k=\sigma+1}^N b_n[k] + \zeta^n \sum_{k=\tau+1}^N b_n[k] \right) \\ &\quad \times \frac{1}{1 + e^{\mp(\ln(z) - \gamma + \sum_{n \geq 1} (\overline{A_n} \zeta^n + A_n \zeta^{-n}))}}, \end{aligned}$$

where the convergence is uniform in ζ varying over the unit circle, and in z varying in a compact subset of \mathbb{C} not touching the negative real semiaxis. Since

$$K(\sigma, x; \tau, y) = \begin{cases} \frac{1}{2\pi i} \oint_{|\zeta|=1} G_+(\zeta) \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ \frac{1}{2\pi i} \oint_{|\zeta|=1} G_-(\zeta) \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma < \tau, \end{cases}$$

this completes the proof of part (i) of Theorem 3.1.

In order to prove Theorem 3.1(ii) we will use Proposition 2.1, which (along with Theorem 3.2) implies

$$\rho_n(\tau_1, x_1; \dots; \tau_n, x_n) = \frac{1}{2\pi i} \oint_{|z|=1} \det[K(\tau_i, x_i; \tau_j, x_j)]_{i,j=1}^n \theta_3(z; t) \frac{dz}{z}.$$

Using the familiar notation $w = \frac{1}{2\pi i} \ln z = \frac{1}{2\pi} \arg z \in [-\frac{1}{2}, \frac{1}{2}]$ and applying the imaginary Jacobi transform, we obtain

$$\theta_3(z; t) = \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{2\pi^2 w^2}{r}} \theta_3\left(e^{\frac{4\pi^2 w}{r}}; e^{-\frac{4\pi^2}{r}}\right)$$

(this formula was already used in the proof of Proposition 3.2 above). The product formula for $\theta_3(e^{\frac{4\pi^2 w}{r}}; e^{-\frac{4\pi^2}{r}})$ implies that this function of w remains uniformly bounded on $[-\frac{1}{2}, \frac{1}{2}]$ as $r \rightarrow 0$, and it uniformly converges to 1 on any open interval inside $[-\frac{1}{2}, \frac{1}{2}]$.

Let us split the domain of integration over z in the formula for $\rho_n(\tau_1, x_1; \dots; \tau_n, x_n)$ above into two arches: $|\arg z| \leq c < \pi$ and $|\arg z| > c$, where $c > 0$ in a arbitrary constant $< \pi$.

The integral over the first arch, by Theorem 3.1(i), is equal to

$$\left(\frac{2\pi}{r}\right)^{\frac{1}{2}} \int_{-\frac{c}{2\pi}}^{\frac{c}{2\pi}} e^{-\frac{2\pi^2 w^2}{r}} \det[\mathcal{K}_{\tau_i, \tau_j}^{(e^{2\pi i w}, \gamma)}(x_i - x_j)]_{i,j=1}^n dw + o(1)$$

as $r \rightarrow 0$. Since $\left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{2\pi^2 w^2}{r}}$ converges to the delta-function at $w = 0$ as $r \rightarrow 0$, the integral above converges to $\det[\mathcal{K}_{\tau_i, \tau_j}^{(1, \gamma)}(x_i - x_j)]_{i,j=1}^n$.

As for the integral over the arch $|\arg z| > c$, we use Proposition 3.2(ii) and the boundedness of the ratio $F(\sigma, \zeta)/F(\tau, \zeta \xi^{-1})$ proved earlier to see that the absolute value of the integrand is bounded by $\text{const} \cdot \epsilon^{-1} r^{-\frac{1}{2}} e^{-\frac{c^2}{2r}}$, which converges to 0 as $r \rightarrow 0$ as long as $\ln \epsilon^{-1} \ll r$. Since this does not contradict our previous assumption that $\epsilon \ll r$, we may ignore the integral over the second arch in the limit $r \rightarrow 0$. This completes the proof of Theorem 3.1. \square

Example 3.4. Consider the periodic Schur process with $N = 1$ and

$$a_n[1] = b_n[1] = \begin{cases} \vartheta, & n = 1, \\ 0, & n > 1, \end{cases}$$

with an arbitrary $\vartheta > 0$. This is equivalent to considering a probability measure on pairs $(\lambda \supset \mu)$ of partitions given by

$$\text{Prob}\{\lambda, \mu\} = \left(e^{\frac{c_1}{1-c_1 c_2}} \prod_{n \geq 1} \frac{1}{1 - (c_1 c_2)^n} \right)^{-1} \frac{\dim^2(\lambda/\mu) c_1^{|\lambda|} c_2^{|\mu|}}{(|\lambda| - |\mu|)!^2},$$

where $c_1 = t\vartheta^2$, $c_2 = \vartheta^{-2}$, and $\dim(\lambda/\mu)$ is the number of standard Young tableaux of shape λ/μ . Here we used the relation

$$s_{\lambda/\mu}(a[1]) = s_{\lambda/\mu}(b[1]) = \frac{\dim(\lambda/\mu) \vartheta^{|\lambda| - |\mu|}}{(|\lambda| - |\mu|)!}$$

and the formula for the partition function from Proposition 1.1.

Theorem 3.1(ii) yields the following limit result for the local correlation functions of this measure: If $x_1(t), \dots, x_n(t) \in \mathbb{Z}'$ are such that

$$\lim_{t \rightarrow 1} (\ln t^{-1} \cdot x_i(t)) = \gamma, \quad i = 1, \dots, n,$$

and the pairwise distances $x_i - x_j$ do not depend on t , then

$$\lim_{t \rightarrow 1} \text{Prob}\{x_1(t), \dots, x_n(t)\} \subset \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1}\} = \det[\mathcal{K}^{(\gamma)}(x_i - x_j)]_{i,j=1}^n,$$

where

$$\mathcal{K}^{(\gamma)}(d) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{1}{1 + e^{\gamma - 2\vartheta \Re \zeta}} \frac{d\zeta}{\zeta^{d+1}}.$$

The formula for the limit correlation functions for μ is exactly the same.

In the limit $\vartheta \rightarrow 0$ the measure on λ 's approaches the uniform measure $\text{Prob}\{\lambda\} = \text{const} \cdot t^{|\lambda|}$ on partitions, and, correspondingly, $\mathcal{K}^{(\gamma)}(d) \rightarrow \delta_{0d} \cdot (1 + e^\gamma)^{-1}$, cf. Comment 4 to Theorem 3.1.

As $\vartheta \rightarrow \infty$, the measure on λ 's looks more like the poissonized Plancherel measure on partitions with weights of the form $\text{Prob}\{\lambda\} \sim \dim^2 \lambda \cdot \text{const}^{|\lambda|}/|\lambda|^2$, because as $c_2 = \vartheta^{-2} \rightarrow 0$, the partition μ tends to be small. When γ is of the same order as ϑ , the local correlation kernel converges to the discrete sine kernel,

$$\lim_{\substack{\vartheta \rightarrow \infty \\ \gamma/\vartheta \rightarrow \Theta}} \mathcal{K}^{(\gamma)}(d) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\mathbf{1}_{2|\Re \zeta| > \Theta} d\zeta}{\zeta^{d+1}} = \begin{cases} 0, & \Theta \geq 2, \\ \frac{\sin(\arccos(\Theta/2) \cdot d)}{\pi d}, & |\Theta| < 2, \\ \delta_{0d}, & \Theta \leq -2, \end{cases}$$

which is exactly the bulk scaling limit of the correlation kernel of the poissonized Plancherel measure with large poissonization parameter, cf. [BOO, Theorem 3].

4. EXTENSIONS OF THE DISCRETE SINE KERNEL

The goal of this section is to construct an infinite-dimensional family of determinantal point processes on $\mathbb{Z} \times \mathbb{Z}$ such that the restrictions of their correlation kernels to copies of \mathbb{Z} obtained by fixing the first coordinate coincide with the discrete sine kernel.

Two such extensions of the discrete sine kernel have been constructed previously: in [OR1] a kernel called the *incomplete beta kernel* was obtained in the bulk limit of the large uniformly distributed plane partitions, and in [BO2] another extension was obtained in the bulk limit of the Markov chains on partitions preserving the Plancherel measure (equivalently, in the bulk of the multi-layer polynuclear growth process with droplet initial conditions). Both these extensions are included in the family that will be constructed below.

We start with some necessary generalities.

Set $\mathfrak{X} = \mathbb{Z} \times \mathbb{Z}$ and let $2^{\mathfrak{X}}$ be the set of all subsets of \mathfrak{X} . The first coordinate of points in \mathfrak{X} will be viewed as “time” while the second coordinate will be viewed as “space”. Pick any exhaustion of \mathfrak{X} by a sequence of increasing finite sets:

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \dots \subset \mathfrak{X}, \quad |\mathcal{B}_n| < \infty, \quad \bigcup_{n \geq 1} \mathcal{B}_n = \mathfrak{X}.$$

For instance, one can take \mathcal{B}_n 's to be growing boxes $\mathcal{B}_n = \{(x, t) : |t| \leq n, |x| \leq n\}$. Clearly, $2^{\mathfrak{X}} = \varprojlim 2^{\mathcal{B}_n}$, and we equip $2^{\mathfrak{X}}$ with the projective limit topology. In other words, a sequence $\{X_n\} \subset 2^{\mathfrak{X}}$ converges to $X \subset \mathfrak{X}$ if and only if for any $m \geq 1$ there exists $M = M(m)$ such that for $n > M$ we have $X_n \cap \mathcal{B}_m = X \cap \mathcal{B}_m$. It is easy to see that the topology does not depend on the choice of \mathcal{B}_n 's. Since projective limits of compact topological spaces are compact, the space $2^{\mathfrak{X}}$ is compact with respect to this topology.

A random point process on \mathfrak{X} is, by definition, a Borel probability measure on $2^{\mathfrak{X}}$.¹³ One way to construct random point processes on \mathfrak{X} is to provide a countable set X_1, X_2, \dots of subsets of \mathfrak{X} and the set of positive weights w_1, w_2, \dots ; $\text{Prob}\{X_i\} = w_i$, satisfying the normalization condition $\sum_{i \geq 1} w_i = 1$.

The correlation functions ρ_n of a random point process on \mathfrak{X} are probabilities for random subsets of \mathfrak{X} to contain a given finite set:

$$\rho_n(t_1, x_1; \dots; t_n, x_n) = \text{Prob}\{X \subset \mathfrak{X} \mid \{(\tau_1, x_1), \dots, (\tau_n, x_n)\} \subset X\}.$$

For a finite set $\mathcal{B} \subset \mathfrak{X}$, the values of the correlation functions on \mathcal{B} define the projection of the measure on $2^{\mathfrak{X}}$ to $2^{\mathcal{B}}$ uniquely. Indeed, the two sets of $2^{|\mathcal{B}|} - 1$ numbers giving for any nontrivial subset \mathcal{B}_0 of \mathcal{B} the probability that the intersection of the random set $X \subset \mathfrak{X}$ with \mathcal{B} contains \mathcal{B}_0 or coincides with \mathcal{B}_0 are related by a nondegenerate linear transformation obtained from the inclusion-exclusion principle. One set consists of the values of the correlation functions while the other one consists of the weights of the subsets with respect to the projected measure on $2^{\mathcal{B}}$. For example, for $\mathcal{B} = \{b_1, b_2\}$ we have

$$\begin{aligned} \text{Prob}\{X \in 2^{\mathfrak{X}} \mid X \cap \mathcal{B} = \{b_1\}\} &= \rho_1(b_1) - \rho_2(b_1, b_2), \\ \text{Prob}\{X \in 2^{\mathfrak{X}} \mid X \cap \mathcal{B} = \{b_2\}\} &= \rho_1(b_2) - \rho_2(b_1, b_2), \\ \text{Prob}\{X \in 2^{\mathfrak{X}} \mid X \cap \mathcal{B} = \{b_1, b_2\}\} &= \rho_2(b_1, b_2). \end{aligned}$$

Lemma 4.1. *Let $\mathcal{P}_1, \mathcal{P}_2, \dots$ be a sequence of random point processes on \mathbb{Z}^2 , and assume that, as $m \rightarrow \infty$, all correlation functions of \mathcal{P}_m converge pointwise: For any $n = 1, 2, \dots$ and any $(\tau_i, x_i) \in \mathbb{Z}^2$, $i = 1, \dots, n$, we have*

$$\lim_{m \rightarrow \infty} \rho_n(\tau_1, x_1; \dots; \tau_n, x_n \mid \mathcal{P}_m) =: r_n(\tau_1, x_1; \dots; \tau_n, x_n).$$

Then there exists a unique random point process \mathcal{P} on \mathbb{Z}^2 such that

$$r_n(\tau_1, x_1; \dots; \tau_n, x_n) = \rho_n(\tau_1, x_1; \dots; \tau_n, x_n \mid \mathcal{P}).$$

Proof. The uniqueness of \mathcal{P} was demonstrated above. The same argument shows that for any finite set \mathcal{B} , the set of possible values of the correlation functions of a probability measure on $2^{\mathcal{B}}$ is described by a finite list of linear inequalities. For instance, in the case $\mathcal{B} = \{b_1, b_2\}$ these inequalities are

$$\begin{aligned} \text{Prob}\{\emptyset\} &= 1 - \rho_1(b_1) - \rho_1(b_2) + \rho_2(b_1, b_2) \geq 0, \\ \text{Prob}\{b_1\} &= \rho_1(b_1) - \rho_2(b_1, b_2) \geq 0, \quad \text{Prob}\{b_2\} = \rho_1(b_2) - \rho_2(b_1, b_2) \geq 0, \\ \text{Prob}\{b_1, b_2\} &= \rho_2(b_1, b_2) \geq 0. \end{aligned}$$

¹³In the conventional terminology, this is actually the definition of a *simple* or *multiplicity free* random point process. Since all our processes are multiplicity free, the definition above is general enough for our purposes.

Clearly, these inequalities are preserved under limit transitions. Hence, for any of the finite sets \mathcal{B}_m used above there exists a unique probability measure \mathcal{P}_m on $2^{\mathcal{B}_m}$ such that its correlation functions coincide with the restrictions of the limit functions r_n to \mathcal{B}_m . Moreover, these measures are consistent: for $m_1 > m_2$ the projection of the measure \mathcal{P}_{m_1} on $2^{\mathcal{B}_{m_1}}$ to $2^{\mathcal{B}_{m_2}}$ coincides with \mathcal{P}_{m_2} . (Indeed, both these measures have the same correlation functions $r_n|_{\mathcal{B}_{m_2}}$.) Since $2^{\mathfrak{X}} = \varprojlim 2^{\mathcal{B}_m}$, we can take $\mathcal{P} = \varprojlim \mathcal{P}_m$. \square

Observe that all that was said above is not specific to the set $\mathfrak{X} = \mathbb{Z}^2$ and applies equally well to any discrete countable set \mathfrak{X} . In particular, we will consider random point processes on $\mathfrak{X}_N := \{1, \dots, N\} \times \mathbb{Z}$ below. Since $\mathfrak{X}_N \subset \mathfrak{X}$, a random point process on \mathfrak{X}_N may also be viewed as a process on \mathfrak{X} whose point configurations are contained in \mathfrak{X}_N almost surely.

Let us now return to the periodic Schur process. So far we have not discussed the positivity of the weights which we used to define the process. One way (but not the only way, see e.g. §8 below) to guarantee the nonnegativity of the weights $W(\lambda, \mu)$ introduced in the beginning of §1 is to take $t > 0$ and to demand that all the specializations of the skew Schur functions are nonnegative. The following classical result is useful.

Due to the Jacobi-Trudi formula

$$s_{\lambda/\mu} = \det[h_{\lambda_i - i - \mu_j + j}]_{i,j=1}^r, \quad r \geq \max\{\ell(\lambda), \ell(\mu)\},$$

see [Macd, §I (5.4)], we need to guarantee the nonnegativity of the determinants in the right-hand side (here h_n 's are the complete homogeneous symmetric functions). Recall that a sequence $\{c_n\}_{n=0}^\infty$ is called *totally positive* if all minors of the matrix $[c_{i-j}]_{i,j \geq 0}$ are nonnegative. Here all c_{-k} for $k > 0$ are assumed to be equal to zero. We will only consider totally positive sequences with $c_0 = 1$; clearly, multiplication of all members of a sequence by the same positive number does not affect total positivity.

The following statement was independently proved by Aissen-Edrei-Schoenberg-Whitney in 1951 [AESW], [Ed], and by Thoma in 1964 [Th]. An excellent exposition of deep relations of this result to representation theory of the infinite symmetric group can be found in Kerov's book [Ke].

Theorem 4.2. *A sequence $\{c_n\}_{n=0}^\infty$, $c_0 = 1$, is totally positive if and only if its generating series has the form*

$$\sum_{n=0}^\infty c_n u^n = e^{\gamma u} \frac{\prod_{i \geq 1} (1 + \beta_i u)}{\prod_{i \geq 1} (1 - \alpha_i u)} =: TP_{\alpha, \beta, \gamma}(u)$$

for certain nonnegative parameters $\{\alpha_i\}$, $\{\beta_i\}$ and γ such that $\sum_i (\alpha_i + \beta_i) < \infty$.

Equivalently, an algebra homomorphism $\phi : \Lambda \rightarrow \mathbb{C}$ takes nonnegative values on all skew Schur functions if and only if the sequence $\{\phi(h_n)\}_{n \geq 0}$ is totally positive, that is,

$$\sum_{n \geq 0} \phi(h_n) u^n = TP_{\alpha, \beta, \gamma}(u)$$

for a suitable choice of parameters (α, β, γ) .

The function $TP_{\alpha,\beta,\gamma}(u)$ is meromorphic in u , and it is holomorphic and nonzero in a small enough neighborhood of the origin. In order to satisfy the convergence conditions, see Remark 2.3, we will actually have to use only the specializations for which $TP_{\alpha,\beta,\gamma}(u)$ is holomorphic and nonzero in a disc of radius greater than 1.

Let us call a specialization a of the algebra of symmetric functions *admissible* if $H(a; u) = TP_{\alpha,\beta,\gamma}(u)$ for certain parameters (α, β, γ) as above, and $\alpha_i, \beta_i < 1$ for all $i \geq 1$.

We say that a specialization a is a union of specializations $a^{(1)}, a^{(2)}, \dots, a^{(m)}$ if

$$H(a; u) = H(a^{(1)}; u) H(a^{(2)}; u) \cdots H(a^{(m)}; u).$$

Note that unions of admissible specializations are admissible. We will use the notation $a = \biguplus_{k=1}^m a^{(k)}$.

Proposition 4.3. *For any $N \geq 1$ and any admissible specializations $a[1], b[1], \dots, a[N], b[N]$ of Λ , denote*

$$a(\sigma, \tau) = \biguplus_{k=\sigma+1}^{\tau} a[k], \quad b(\sigma, \tau) = \biguplus_{k=\sigma+1}^{\tau} b[k], \quad a = a(0, N], \quad b = b(0, N].$$

Assume that $a = b$. Then for any $C > 0$ there exists a unique random point process on $\{1, \dots, N\} \times \mathbb{Z}$ with determinantal correlation functions

$$\rho_n(\tau_1, x_1; \dots; \tau_n, x_n) = \det[\mathcal{K}_{\tau_i, \tau_j}(x_i - x_j)]_{i,j=1}^n$$

(here $n \geq 1$, $\tau_i \in \{1, \dots, N\}$, $x_i \in \mathbb{Z}$ are arbitrary), and the correlation kernel

$$\mathcal{K}_{\sigma, \tau}(d) = \begin{cases} \frac{1}{2\pi i} \oint_{|z|=1} \frac{(H(a(\sigma, \tau]; \zeta^{-1}) H(b(\sigma, \tau]; \zeta))^{-1}}{1 + C (H(a; \zeta^{-1}) H(b; \zeta))^{-1}} \frac{d\zeta}{\zeta^{d+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \oint_{|z|=1} \frac{H(a(\tau, \sigma]; \zeta^{-1}) H(b(\tau, \sigma]; \zeta)}{1 + C^{-1} H(a; \zeta^{-1}) H(b; \zeta)} \frac{d\zeta}{\zeta^{d+1}}, & \sigma > \tau. \end{cases}$$

Proof. Consider the periodic Schur process with $t \in (0, 1)$ and specializations $a[1], b[1], \dots, a[N], b[N]$ as in the hypothesis. By Theorem 4.2, all weights of this measure on \mathbb{Y}^N are nonnegative. Mapping \mathbb{Y}^N to $2^{\{1, \dots, N\} \times \mathbb{Z}'}$ via

$$(\lambda^{(1)}, \dots, \lambda^{(N)}) \mapsto \left(\left\{ \lambda_i^{(1)} - i + \frac{1}{2} \right\}_{i \geq 1}, \dots, \left\{ \lambda_i^{(N)} - i + \frac{1}{2} \right\}_{i \geq 1} \right)$$

yields a random point process on $\{1, \dots, N\} \times \mathbb{Z}'$ whose correlation functions were computed in §2. Shifting the space variable $x \mapsto \lfloor \ln C / \ln t^{-1} \rfloor + x + \frac{1}{2}$ and using Theorem 3.1(ii) we see that the correlation functions of our process converge, as $t \rightarrow 1-$, to those given in the hypothesis above. Lemma 4.1 completes the proof. \square

The main result of this section is the following statement.

Theorem 4.4. *For any doubly infinite sequences $\{a[k], b[k]\}_{k \in \mathbb{Z}}$ of admissible specializations of Λ and any $c \in (0, \pi)$, there exists a determinantal¹⁴ point process on $\mathbb{Z} \times \mathbb{Z}$ with the correlation kernel*

$$\mathcal{K}_{\sigma, \tau}(x - y) = \begin{cases} \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} (H(a(\sigma, \tau]; \zeta^{-1}) H(b(\sigma, \tau]; \zeta))^{-1} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int_{e^{ic}}^{e^{-ic}} H(a(\tau, \sigma]; \zeta^{-1}) H(b(\tau, \sigma]; \zeta) \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma > \tau, \end{cases}$$

where the integrals are taken over positively oriented arches of the unit circle.

Comments. 1. The equal time values of the kernel above are exactly those of the discrete sine kernel on \mathbb{Z} ; for any $\tau \in \mathbb{Z}$

$$\mathcal{K}_{\tau, \tau}(x - y) = \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} \frac{d\zeta}{\zeta^{x-y+1}} = \frac{e^{ic(x-y)} - e^{-ic(x-y)}}{2\pi i(x-y)} = \frac{\sin(c(x-y))}{\pi(x-y)}.$$

In this sense the kernels $\mathcal{K}_{\sigma, \tau}(x - y)$ provide extensions of the discrete sine kernel.

2. The point processes in questions are clearly invariant with respect to the shifts of the space coordinate. If one wants the processes to be invariant with respect to the time shifts as well, one has to take all specializations $a[k]$ to be the same, and all specializations $b[k]$ to be the same.

3. Using Lemma 4.1 we can slightly relax the conditions on the specializations $a[k]$, $b[k]$ by allowing the parameters α_i to be equal to 1. (Recall that they are required to be strictly less than 1 by the definition of admissible specializations.) Taking trivial specializations $a[k]$ and choosing $b[k]$ so that $H(b[k]; u) = (1 - u)^{-1}$, we obtain the kernel

$$\mathcal{K}_{\sigma, \tau}(x - y) = \begin{cases} \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} (1 - \zeta)^{\tau-\sigma} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int_{e^{ic}}^{e^{-ic}} (1 - \zeta)^{\tau-\sigma} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma > \tau, \end{cases}$$

which is the incomplete beta kernel of [OR1]. We will also see this kernel arising in the bulk limit of the cylindric partitions in the subsequent sections.

4. The choice $H(a[k]; u) = H(b[k]; u) = e^{\text{const}_k \cdot u}$ yields the extension of the discrete sine kernel obtained in [BO2, Theorem 4.2].

Proof. It suffices to prove the statement for the processes on $\{1, \dots, N\} \times \mathbb{Z}$ and finite sequences $\{a[k], b[k]\}_{k=1}^N$ of admissible specializations. Indeed, then one can just embed such processes in $2^{\mathbb{Z} \times \mathbb{Z}}$ and take the limit as $N \rightarrow \infty$ using Lemma 4.1.

In the finite N case we use Proposition 4.3 to construct a process on $\{1, \dots, 2N+1\} \times \mathbb{Z}$ with extra $2(N+1)$ specializations $a[N+1], b[N+1], \dots, a[2N+1], b[2N+1]$ defined by

$$\begin{aligned} a[N+k] &= b[k], & b[N+k] &= a[k], & k &= 1, \dots, N, \\ H(a[2N+1]; u) &= H(b[2N+1]; u) = e^{Mu}, & M &> 0. \end{aligned}$$

¹⁴i.e., with determinantal correlation functions.

Then

$$\biguplus_{k=1}^{2N+1} a[k] = a = b = \biguplus_{k=1}^{2N+1} b[k],$$

$$H(a[2N+1]; \zeta^{-1})H(b[2N+1]; \zeta) = e^{2M\Re\zeta}.$$

Choosing the constant C in Proposition 4.3 to be $C = e^{2M \cos c}$, for ζ on the unit circle we obtain

$$\lim_{M \rightarrow \infty} \frac{1}{1 + C(H(a; \zeta^{-1})H(b; \zeta))^{-1}} = \begin{cases} 1, & \Re\zeta > \cos c, \\ 0, & \Re\zeta < \cos c, \end{cases}$$

$$\lim_{M \rightarrow \infty} \frac{1}{1 + C^{-1}H(a; \zeta^{-1})H(b; \zeta)} = \begin{cases} 0, & \Re\zeta > \cos c, \\ 1, & \Re\zeta < \cos c. \end{cases}$$

Hence, as $M \rightarrow \infty$ we have the convergence of the correlation functions of our M -dependent processes restricted to $\{1, \dots, N\} \times \mathbb{Z}$ to the needed values, and Lemma 4.1 completes the proof. \square

5. CYLINDRIC PARTITIONS

Cylindric partitions were first introduced by I. Gessel and C. Krattenthaler in [GK]. We will initially follow their paper in our exposition.

Let λ and μ be two partitions. Assume that $\lambda \supset \mu$ and denote the length (=number of nonzero parts) of λ by l . A *plane partition of shape* λ/μ is a planar array π of integers of the form

$$\begin{array}{cccccccc} & & & & \pi_{1,\mu_1+1} & \cdots & \pi_{1,\lambda_l} & \cdots & \cdots & \pi_{1,\lambda_1} \\ & & & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots & \pi_{2,\lambda_l} & \cdots & \pi_{2,\lambda_2} \\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{l,\mu_l+1} & \cdots & \pi_{l,\mu_2+1} & \cdots & \pi_{l,\mu_1+1} & \cdots & \pi_{l,\lambda_l} & & & \end{array}$$

such that the rows and columns are weakly decreasing: $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}$.

A *cylindric partition of shape* $\lambda/\mu/d$ can be viewed as a plane partition with an additional relation between the first and the last rows. This relation depends on an integral parameter d and it has the form $\pi_{l,j} \geq \pi_{1,j+d}$ for all j . In other words, a cylindric partition has to remain a plane partition when the last row shifted by d to the right is placed on top of the first row. Here is an example of a cylindric partition of shape $(8, 6, 3)/(3, 1)/4$. On the second picture the shifted last row (in bold) is placed on top of the first one.

$$\begin{array}{cccccc} & & 7 & 5 & 2 & 1 \\ 10 & 10 & 6 & 5 & 1 & 1 \\ 11 & 9 & 1 & & & \end{array} \qquad \begin{array}{cccccc} & & \mathbf{11} & \mathbf{9} & \mathbf{1} & \\ & & 7 & 5 & 2 & 1 \\ 10 & 10 & 6 & 5 & 1 & 1 \\ 11 & 9 & 1 & & & \end{array}$$

Note that d has to be greater or equal to μ_1 .

The *norm* $|\pi|$ of the cylindric partition π is the sum of its elements. In the example above $|\pi| = 68$.

For our purposes it is more convenient to parameterize cylindric partitions differently. Namely, let us read them along the lines with fixed content $j - i$ (these

lines are parallel to the diagonal which has content zero). On each line we observe an ordinary partition, and partitions on neighboring lines are different by adding or removing a horizontal strip.¹⁵ The necessary number of fixed content lines to be taken into account is equal to $N := l + d$, and the content can be considered as an element of the cyclic group $\mathbb{Z}/N\mathbb{Z}$ of order N .

The example above leads to the following (periodic with period $N = 7$) sequence of partitions:

$$\dots \prec (11, 2, 1) \succ (9, 1) \prec (10, 1) \succ (10) \succ (6) \prec (7, 5) \succ (5, 1) \prec (11, 2, 1) \succ \dots$$

The “ \prec ” or “ \succ ” relation of the neighboring partitions depends on the boundary of the Young diagram μ : horizontal edges correspond to “ \succ ” and vertical edges correspond to “ \prec ”.

It is impossible to reconstruct λ and μ from such a sequence of partitions: in the example above if we remove the last row from the second picture then the resulting partition sequence will be the same while we would be looking at a cylindric partition of type $(6, 6, 6)/(3, 2)/4$. However, d and l remain invariant — d is equal to the total number of “ \prec ” and l is equal to the total number of “ \succ ” in a period.

Let us encode the sequence of “ \succ ” and “ \prec ” by assigning to a cylindric partition a periodic sequence of 1’s and -1 ’s; 1’s correspond to \succ ’s and -1 ’s correspond to \prec ’s. Thus, the example above produces $(\dots, -1, 1, -1, 1, 1, -1, 1, -1, 1, \dots)$ (this sequence is periodic with period $N = 7$). We will call this sequence the *profile* of the corresponding cylindric partition. Clearly, the profile depends only on μ, d , and l , but not on λ .

We say that the profile is *marked* if there is a marked -1 in each period, and the distance between any marked -1 ’s is a multiple of the period. In other words, if we consider profiles as maps from $\mathbb{Z}/N\mathbb{Z}$ to $\{-1, 1\}$ then marking corresponds to choosing a distinguished element of $\mathbb{Z}/N\mathbb{Z}$ in the preimage of -1 .

In order to associate a *marked* profile to any cylindric partition we will mark the -1 ’s corresponding to the relation of partitions in lines with content d and $-l + 1$. Thus, in our example the marked -1 ’s corresponds to $(5, 1) \subset (11, 9, 1)$ and the marked profile is

$$(\dots, -1^*, 1, -1, 1, 1, -1, 1, -1^*, 1, \dots).$$

Marked profiles are in one-to-one correspondence with triples (μ, d, l) with $d \geq \mu_1$ and $l \geq \ell(\mu)$.

Let us associate to any marked profile two periodic sequences $A[k]$, $B[k]$ with period N and elements 0 or 1 such that the difference $A[k] - B[k]$ gives the element of the profile with distance k from the marked -1 . In the example above

$$\begin{aligned} A &= (\dots, 0^*, 1, 0, 1, 1, 0, 1, 0^*, 1, \dots), \\ B &= (\dots, 1^*, 0, 1, 0, 0, 1, 0, 1^*, 0, \dots), \end{aligned}$$

where we marked $A[0 \bmod N]$ and $B[0 \bmod N]$. As was mentioned above, we have

$$d = \sum_{k=1}^N A[k], \quad l = \sum_{k=1}^N B[k].$$

¹⁵ Recall that a Young diagram (equivalently, a partition) ν can be obtained from another Young diagram κ by adding a horizontal strip (notation $\kappa \prec \nu$ or $\nu \succ \kappa$) if and only if $\nu_i \geq \kappa_i \geq \nu_{i+1}$ for all $i \geq 1$.

For any integer m let $m(N)$ be the smallest positive integer such that

$$m \equiv m(N) \pmod{N}.$$

For instance, $1(N) = 1$ and $-1(N) = N - 1$.

In what follows we will be interested in probability measures on cylindric partitions π with fixed μ, d , and l (equivalently, fixed marked profile), whose weights are proportional to $s^{|\pi|}$, where $s \in (0, 1)$ is a parameter. We will call these probability measures *uniform* as the weights of the cylindric partitions with the same norm are equal. The next statement provides the partition function for the weights $s^{|\pi|}$.

Proposition 5.1. *For any partition μ and integers $d \geq \mu_1, l \geq \ell(\mu)$, the following identity holds:*

$$\sum_{\{\lambda: \ell(\lambda) \leq l\}} \sum_{\{\pi: \pi \text{ has shape } \lambda/\mu/d\}} s^{|\pi|} = \prod_{n \geq 1} \frac{1}{1 - s^{nN}} \prod_{\substack{p \in \overline{1, N}: A[p]=1 \\ q \in \overline{1, N}: B[q]=1}} \frac{1}{1 - s^{(p-q)(N) + (n-1)N}},$$

where $N = d + l$, and $A[k]$ and $B[k]$ are sequences of 0's and 1's associated to (μ, d, l) as described above.

Comments. 1. It is convenient to view the function $(p - q)(N)$ as an array on the $N \times N$ torus with rows and columns parameterized by p and q :

$$\begin{bmatrix} N & N-1 & N-2 & \dots & 3 & 2 & 1 \\ 1 & N & N-1 & \dots & 4 & 3 & 2 \\ 2 & 1 & N & \dots & 5 & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ N-2 & N-3 & N-4 & \dots & 1 & N & N-1 \\ N-1 & N-2 & N-3 & \dots & 2 & 1 & N \end{bmatrix}$$

In the formula above we choose rows p such that $A[p] = 1$ and columns q such that $B[q] = 1$. Note that the sets of rows and columns chosen in such a way form a disjoint splitting of $\{1, \dots, N\}$ into two sets; the first set contains d elements and the second set contains l elements. Hence, the total number of factors in the second product equals dl .

2. The right-hand side of the formula for the partition function depends on the profile of (μ, d, l) , but not on the marked profile. This is in agreement with rotational symmetry of the problem. The sum in the left-hand side of the formula may be viewed as the sum over all cylindric partitions with fixed marked profile.

3. If $d = 0$ then μ must be empty, and all rows of the corresponding cylindric partitions become identical. Accordingly, the second product in the right-hand side disappears, and we recover the formula for the partition function of the weights $s^{(\# \text{ of boxes}) \cdot N}$ on ordinary partitions.

4. If $d, l \gg 1$ and $A = (1, 1, \dots, 1, 0, 0, \dots, 0^*)$, $B = (0, 0, \dots, 0, 1, 1, \dots, 1^*)$ then $\mu = d^l$ and the corresponding cylindric partitions of small enough norm are in one-to-one correspondence with similar plane partitions. Accordingly, looking at the top right corner of the matrix in Comment 1 above, we see that the partition function for small powers of s looks like that for the plane partitions: $\prod_{n \geq 1} (1 - s^n)^{-n}$. In the

limit $d, l \rightarrow \infty$ one recovers the celebrated MacMahon's formula for the partition function of the weights s^{norm} on plane partitions.

5. The case of largest rotational symmetry corresponds to $A = (1, 0, 1, 0, \dots, 1, 0^*)$ and $B = (0, 1, 0, 1, \dots, 0, 1^*)$. Then $l = d$, $N = 2d$, and $\mu = (d, d-1, \dots, 2, 1)$ is the staircase partition. In this case the formula for the partition function slightly simplifies to give

$$\prod_{n \geq 1} \frac{1}{1 - s^{2nd}} \prod_{m=1}^d \frac{1}{(1 - s^{2m-1+2(n-1)d})^d}.$$

Proof. It is well known that the skew Schur function $s_{\lambda/\mu}$ specialized at a single variable x is nonzero if and only if $\lambda \succ \mu$, in which case $s_{\lambda/\mu}(x) = x^{|\lambda| - |\mu|}$.

This observation immediately implies that cylindric partitions in question are in one-to-one correspondence with trajectories of the periodic Schur process with period N determined by the specializations (see §1 for notations)

$$\tilde{a}_n[k] = \frac{1}{n} A[k], \quad \tilde{b}_n[k] = \frac{1}{n} B[k], \quad k = 1, \dots, N,$$

or, in different terms,

$$H(a[k]; u) = (1 - s^k u)^{-A[k]}, \quad H(b[k]; u) = (1 - s^{-k} u)^{-B[k]}, \quad k = 1, \dots, N.$$

Namely, we read the cylindric partitions along the lines with fixed content, and the resulting ordinary partitions form a periodic Schur process. Recall that following the marking rule we denote the partition coming from the line with content $(m \bmod N)$ by $\lambda^{(m+l)}$. In particular, the marked place of the profile corresponds to the relation $\lambda^{(0)} \prec \lambda^{(1)}$.

The formula of Proposition 5.1 is then a direct corollary of Proposition 1.1. \square

The one-to-one correspondence of the cylindric partitions with a given profile and trajectories of the periodic Schur process explained in the proof above also implies that we can use the results of §2 to obtain the correlation functions of the uniform measure on cylindric partitions.

We define the correlation functions of the uniform measure on cylindric partitions with a given profile as the dynamical correlation functions of the corresponding periodic Schur process.

Proposition 5.2. *The correlation functions of the uniform measure on cylindric partitions with a given profile $\{A[k], B[k]\}_{k=1}^N$ have the following form: For any $n \geq 1$, $\tau_1, \dots, \tau_n \in \{1, \dots, N\}$, $x_1, \dots, x_n \in \mathbb{Z}'$,*

$$\rho_n(\tau_1, x_1; \dots; \tau_n, x_n) = \oint_{|z|=1} \det[K(\tau_i, x_i; \tau_j, x_j)]_{i,j=1}^n \frac{\theta_3(z; t) dz}{z},$$

where $t = s^N$,

$$K(\sigma, x; \tau, y) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)^2} \oint_{\zeta} \oint_{\eta} \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} \frac{\theta_3(z\zeta\eta; t)}{\theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)} \frac{d\zeta d\eta}{\zeta^{x+\frac{1}{2}} \eta^{y+\frac{1}{2}}},$$

the integrals are taken over circles centered at the origin with radii satisfying

$$s^{\sigma+1} < |\zeta| < s^{\sigma}, \quad s^{-\tau} < |\eta| < s^{-\tau-1}, \quad |\zeta\eta| > 1 \quad \text{if } \sigma = \tau,$$

and

$$F(\tau, \zeta) = \exp \sum_{n \geq 1} \frac{1}{n(1 - s^{nN})} \left(\sum_{k=1}^{\tau} B[k](s^{-k}\zeta)^n + \sum_{k=\tau+1}^N B[k](s^{N-k}\zeta)^n - \sum_{k=1}^{\tau} A[k](s^{k+N}/\zeta)^n - \sum_{k=\tau+1}^N A[k](s^k/\zeta)^n \right).$$

Proof. The statement follows from Theorem 2.2, Proposition 2.1, and Remark 2.4. The only new effect that we have to be careful about is the choice of the contours of integration, because the function $F(\tau, \zeta)$ is correctly defined by the formula above only in the ring $s^{\tau+1} < |\zeta| < s^{\tau}$, which explains our conditions on the contours. \square

Remark 5.3. The integration over z in the formula for the correlation functions above can be carried out explicitly to produce a multivariate integral formula similar to that in Corollary 2.8.

Remark 5.4. The function $F(\tau, \zeta)$ can be written in terms of infinite products:

$$F(\tau, \zeta) = \frac{\prod_{k \in \overline{1, \tau}: A[k]=1} (s^{k+N}/\zeta; t)_{\infty} \prod_{k \in \overline{\tau+1, N}: A[k]=1} (s^k/\zeta; t)_{\infty}}{\prod_{k \in \overline{1, \tau}: B[k]=1} (s^{-k}\zeta; t)_{\infty} \prod_{k \in \overline{\tau+1, N}: B[k]=1} (s^{N-k}\zeta; t)_{\infty}},$$

where, as usual, $(a; t)_{\infty} = \prod_{n \geq 1} (1 - at^n)$. This formula also provides the analytic continuation of $F(\tau, \zeta)$ from the ring $s^{\tau+1} < |\zeta| < s^{\tau}$.

6. BULK OF LARGE CYLINDRIC PARTITIONS WITH FINITE OR SLOWLY GROWING PERIOD

In this section we compute the local limit of the correlation functions of the uniform measure on cylindric partitions with fixed profile $\{A[k], B[k]\}_{k=1}^N$ in two cases: when N is fixed and the profile does not depend on the small parameter $r = \ln t^{-1} = N \ln s^{-1}$, and when N is growing in such a way that r still tends to 0 as $s \rightarrow 1-$.

In both cases an important role is played by the parameter

$$\varkappa = \frac{l}{d} = \frac{\sum_{k=1}^N B[k]}{\sum_{k=1}^N A[k]}$$

which we call the *slope* of the profile. In the case of growing N in order to have a limit we will need to assume that the slope tends to a limiting value strictly between 0 and ∞ .

We need to do some preliminary work before stating the results.

For any $\varkappa > 0$, let Γ_{\varkappa} be the contour in the complex plane defined by

$$\begin{aligned} \Gamma_{\varkappa} &= \left\{ z = 1 - \frac{\sin(1+\varkappa)\phi}{\sin \varkappa\phi} e^{i\phi} \mid \phi \in \left(-\frac{\pi}{1+\varkappa}, \frac{\pi}{1+\varkappa}\right) \right\} \\ &= \left\{ -\frac{\sin \frac{\varphi}{1+\varkappa}}{\sin \frac{\varphi}{1+\varkappa^{-1}}} e^{i\varphi} \mid \varphi \in (-\pi, \pi) \right\}. \end{aligned}$$

This is a piecewise smooth closed curve which has a corner-like singularity at $z = 1$ as $\phi \rightarrow \pm \frac{\pi}{1+\varkappa}$ or $\varphi \rightarrow \pm \pi$. The origin is located inside Γ_\varkappa . The curve is also symmetric with respect to the real axis, it intersects the negative semiaxis at the point $z = -\varkappa^{-1}$, and Γ_{\varkappa_1} is located inside Γ_{\varkappa_2} if $\varkappa_1 > \varkappa_2$. Note that Γ_1 is the unit circle, which is the only case when the curve is smooth at $z = 1$. Whenever we use Γ_\varkappa as the integration contour, we will assume that it is oriented counterclockwise.

Our interest in the family $\{\Gamma_\varkappa\}$ is explained by the following statement. We use the principal branch of the logarithm function below.

Lemma 6.1. *For any $\varkappa > 0$, set $f_\varkappa(z) = \varkappa \ln(1 - z) + \ln(1 - z^{-1})$. This is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, and*

$$\{z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0} \mid f_\varkappa(z) \in \mathbb{R}\} = \mathbb{R}_{< 0} \cup \Gamma_\varkappa.$$

Further, $\Im f_\varkappa(z) < 0$ if and only if z is located inside Γ_\varkappa and below the real axis or outside Γ_\varkappa and above the real axis.

On the curve Γ_\varkappa the function f_\varkappa equals

$$f_\varkappa\left(1 - \frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi} e^{i\phi}\right) = \varkappa \ln\left(\frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi}\right) + \ln\left(\frac{\sin(\varkappa+1)\phi}{\sin \phi}\right).$$

This function increases on $(-\frac{\pi}{\varkappa+1}, 0)$, decreases on $(0, \frac{\pi}{\varkappa+1})$, and its maximal value on Γ_\varkappa equals

$$f_\varkappa(-\varkappa^{-1}) = \ln(1 + \varkappa) + \varkappa \ln(1 + \varkappa^{-1}).$$

Proof. We have

$$\frac{df_\varkappa(re^{i\varphi})}{dr} = r^{-1} \left(\varkappa - \frac{1 + \varkappa}{1 - re^{i\varphi}} \right) = r^{-1} \left(\varkappa - \frac{(1 + \varkappa)(1 - re^{-i\varphi})}{|1 - re^{i\varphi}|^2} \right),$$

whence on any ray $\{z = re^{i\varphi} \mid r > 0\}$, the function $\Im f_\varkappa(z)$ is strictly decreasing as a function of r for $\varphi \in (0, \pi)$, and it is strictly increasing for $\varphi \in (-\pi, 0)$. Thus, on any such ray there cannot be more than one point z such that $f_\varkappa(z) \in \mathbb{R}$.

On the other hand, for $z = 1 - \frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi} e^{i\phi} \in \Gamma_\varkappa$ we compute

$$z = -\frac{\sin \phi}{\sin \varkappa\phi} e^{i(\varkappa+1)\phi}, \quad z^{-1} = -\frac{\sin \varkappa\phi}{\sin \phi} e^{-i(\varkappa+1)\phi}, \quad 1 - z^{-1} = \frac{\sin(\varkappa+1)\phi}{\sin \phi} e^{-i\varkappa\phi},$$

which implies the first statement of the lemma and the formula for f_\varkappa on the curve. Since

$$\begin{aligned} \left(\frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi} \right)' &= \frac{\sin(2\varkappa+1)\phi - (2\varkappa+1)\sin \phi}{2(\sin \varkappa\phi)^2} < 0, \\ \left(\frac{\sin(\varkappa+1)\phi}{\sin \phi} \right)' &= \frac{\varkappa \sin(2\varkappa+1)\phi - (2\varkappa+1)\sin \varkappa\phi}{2(\sin \phi)^2} < 0, \end{aligned}$$

for $\phi \in (0, \frac{\pi}{\varkappa+1})$, and f_\varkappa restricted to Γ_\varkappa is an even function of ϕ , the last claim of the lemma also follows. \square

For any profile $\{A[k], B[k]\}$ denote

$$A(\sigma, \tau) = \sum_{k=\sigma+1}^{\tau} A[k], \quad B(\sigma, \tau) = \sum_{k=\sigma+1}^{\tau} B[k].$$

Then $d = A(0, N]$, $l = B(0, N]$, and $\varkappa = B(0, N]/A(0, N]$.

Now we can state the main result of this section.

Theorem 6.2. *In the limit $s \rightarrow 1-$, the correlation functions of the uniform measure on cylindric partitions with a given profile $\{A[k], B[k]\}_{k=1}^N$ have a limit in the following sense: Choose $x_1(s), \dots, x_n(s) \in \mathbb{Z} + \frac{1}{2}$ such that as $s \rightarrow 1-$, $rx_k(s) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j = x_i(s) - x_j(s)$ remain constant. Then for any $\tau_1, \dots, \tau_n \in \{1, \dots, N\}$*

$$\lim_{s \rightarrow 1-} \rho_n(\tau_1, x_1(s); \dots, \tau_n, x_n(s)) = \det[\mathcal{K}_{\tau_i, \tau_j}^{(\gamma)}(x_i - x_j)]_{i,j=1}^n,$$

where the correlation kernel has the form

$$\mathcal{K}_{\sigma, \tau}^{(\gamma)}(x - y) = \begin{cases} \frac{1}{2\pi i} \oint_{\Gamma_\varkappa} \frac{(1 - \zeta)^{B(\sigma, \tau]} (1 - \zeta^{-1})^{A(\sigma, \tau]}}{1 + e^\gamma (1 - \zeta)^l (1 - \zeta^{-1})^d} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \oint_{\Gamma_\varkappa} \frac{(1 - \zeta)^{-B(\tau, \sigma]} (1 - \zeta^{-1})^{-A(\tau, \sigma]}}{1 + e^{-\gamma} (1 - \zeta)^{-l} (1 - \zeta^{-1})^{-d}} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma > \tau. \end{cases}$$

Comments. 1. By Lemma 6.1, the expression $(1 - \zeta)^l (1 - \zeta^{-1})^d$ is nonnegative on Γ_\varkappa , which shows that the kernel $\mathcal{K}_{\sigma, \tau}^{(\gamma)}$ is correctly defined for $\sigma \leq \tau$. On the other hand, for $\sigma > \tau$ we can rewrite the formula for the kernel as

$$\mathcal{K}_{\sigma, \tau}^{(\gamma)}(x - y) = -\frac{1}{2\pi i} \oint_{\Gamma_\varkappa} \frac{e^\gamma (1 - \zeta)^{l-B(\tau, \sigma]} (1 - \zeta^{-1})^{d-A(\tau, \sigma]}}{1 + e^\gamma (1 - \zeta)^l (1 - \zeta^{-1})^d} \frac{d\zeta}{\zeta^{x-y+1}},$$

and this integral also always makes sense.

2. The global limit density function

$$\rho(\gamma) = \mathcal{K}_{\tau, \tau}^{(\gamma)}(0) = \frac{1}{2\pi i} \oint_{\Gamma_\varkappa} \frac{1}{1 + e^\gamma (1 - \zeta)^l (1 - \zeta^{-1})^d} \frac{d\zeta}{\zeta}$$

does not depend on τ , which means that it is invariant with respect to rotations of the cylindric partitions. Note also that $\rho(\gamma)$ depends on the fixed profile only through $d = A(0, N]$ and $l = B(0, N]$, or, in other words, through the period $N = d + l$ and the slope $\varkappa = l/d$.

3. The simplest nontrivial (i.e., not coinciding with the uniform measure on ordinary partitions) example is $d = l = 1$. Then Γ_\varkappa is the unit circle, and it is not hard to evaluate the global limit density function and the corresponding hypothetical limit shape (see Comment 3 after Theorem 3.1 for explanations) explicitly:

$$\rho(\gamma) = \frac{1}{\sqrt{1 + 4e^\gamma}}, \quad v(u) = u + 4 \operatorname{arctanh}(\sqrt{1 + 4e^u}) + \frac{i\pi}{2}.$$

For larger values of d and l explicit integration is also possible but as d and l grows it becomes increasingly tedious.

4. A formal application of Theorem 3.1 to the uniform measure on cylindric partitions produces the correct integrand in the formula for the limit kernel, but the integration contour is different unless $\varkappa = 1$. For $\varkappa \neq 1$ using the unit circle (as Theorem 3.1 suggests) instead of the curve Γ_\varkappa may lead to a wrong answer!

Theorem 6.3. *In the limit*

$$s \rightarrow 1-, \quad d \rightarrow \infty, \quad l \rightarrow \infty, \quad r = -N \ln s \rightarrow 0, \quad l/d \rightarrow \varkappa > 0,$$

the correlation functions of the uniform measure on cylindric partitions with a given profile $\{A[k], B[k]\}_{k=1}^N$ have a limit in the following sense: Choose $x_1(s), \dots, x_n(s) \in \mathbb{Z} + \frac{1}{2}$ such that as $s \rightarrow 1-$, $rx_k(s)/N \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j = x_i(s) - x_j(s)$ remain constant. Also choose the time moments $\tau_1(s), \dots, \tau_n(s)$ such that as $s \rightarrow 1$ the pairwise distances $\tau_i(s) - \tau_j(s)$ remain uniformly bounded. Then

$$\rho_n(\tau_1(s), x_1(s); \dots, \tau_n(s), x_n(s)) = \det[\mathcal{K}_{\tau_i(s), \tau_j(s)}^{(\gamma)}(x_i - x_j)]_{i,j=1}^n + o(1),$$

where the estimate is uniform over any set of profiles with the slope uniformly convergent to \varkappa as $s \rightarrow 1$, and the correlation kernel has the following form:

$$\mathcal{K}_{\tau, \tau}^{(\gamma)}(x - y) = \delta_{x-y, 0} \quad \text{if} \quad \gamma \leq \gamma_0(\varkappa) := -\frac{\ln(1 + \varkappa)}{1 + \varkappa} - \frac{\ln(1 + \varkappa^{-1})}{1 + \varkappa^{-1}},$$

and for $\gamma > \gamma_0(\varkappa)$, using the notation $\zeta(\phi) = 1 - \frac{\sin(1+\varkappa)\phi}{\sin \varkappa\phi} e^{i\phi}$, we have

$$\mathcal{K}_{\sigma, \tau}^{(\gamma)}(x - y) = \begin{cases} \frac{1}{2\pi i} \int_{\zeta(\phi)}^{\overline{\zeta(\phi)}} (1 - \zeta)^{B(\sigma, \tau)} (1 - \zeta^{-1})^{A(\sigma, \tau)} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int_{\zeta(\phi)}^{\zeta(\phi)} (1 - \zeta)^{-B(\tau, \sigma)} (1 - \zeta^{-1})^{-A(\tau, \sigma)} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma > \tau, \end{cases}$$

where both integration contours leave the origin on their left sides, and the constant $\phi = \phi(\gamma) \in (0, \frac{\pi}{1+\varkappa})$ from the limits of the integrals above is uniquely determined by the relation

$$-(1 + \varkappa)\gamma = \varkappa \ln \left(\frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi} \right) + \ln \left(\frac{\sin(\varkappa+1)\phi}{\sin \phi} \right).$$

Comments. 1. The existence of the unique $\phi(\gamma)$ for $\gamma > \gamma_0$, as well as nonexistence of ϕ satisfying the relation above for $\gamma < \gamma_0$, follows directly from Lemma 6.1. Using this lemma it is also easy to see that the statement of Theorem 6.3 can be formally deduced from Theorem 6.2. Indeed,

$$\begin{aligned} \ln(e^\gamma(1 - \zeta)^l(1 - \zeta^{-1})^d) &= N \left(\frac{\gamma}{N} + \frac{\varkappa}{1+\varkappa} \ln(1 - \zeta) + \frac{1}{1+\varkappa} \ln(1 - \zeta^{-1}) \right) \\ &= \frac{N}{1+\varkappa} \left(\frac{(1+\varkappa)}{N} \gamma + f_\varkappa(\zeta) \right). \end{aligned}$$

Thus, for $\zeta \in \Gamma_\varkappa$, $N \rightarrow \infty$, $\gamma \rightarrow \infty$, and $\gamma/N \rightarrow \hat{\gamma}$ we have

$$\lim_{N \rightarrow \infty} e^\gamma(1 - \zeta)^l(1 - \zeta^{-1})^d = \begin{cases} 0, & f_\varkappa(\zeta) < -(1 + \varkappa)\hat{\gamma}, \\ +\infty, & f_\varkappa(\zeta) > -(1 + \varkappa)\hat{\gamma}. \end{cases}$$

Substituting this limit relation in the formulas of Theorem 6.2 we recover the formulas of Theorem 6.3.

2. The global limit density function for $\gamma < \gamma_0(\varkappa)$ is identically equal to 1, while for $\gamma > \gamma_0(\varkappa)$ it is equal to

$$\rho(\gamma) = \mathcal{K}_{\tau,\tau}^{(\gamma)} = \frac{1}{2\pi i} \int_{\zeta(\phi)}^{\overline{\zeta(\phi)}} \frac{d\zeta}{\zeta} = -\frac{\arg \zeta(\phi)}{\pi} = 1 - \frac{(1+\varkappa)\phi}{\pi}.$$

Note that this expression is independent of τ , which reflects the rotational invariance of the limit density. As $\gamma \rightarrow \gamma_0$ we have $\phi(\gamma) \rightarrow 0$ and $\rho(\gamma) \rightarrow 1$. On the other hand, as $\gamma \rightarrow +\infty$ we have $\phi(\gamma) \rightarrow \frac{\pi}{1+\varkappa}$ and $\rho(\gamma) \rightarrow 0$.

The fact that $\rho(\gamma) \equiv 1$ for $\gamma < \gamma_0$ means that the random Young diagrams in question have the lower edge — the event of having columns of length substantially greater than $N\gamma_0(\varkappa)/r$ has vanishing probability.

3. For $\varkappa = 1$ the formulas simplify, and we obtain

$$\begin{aligned} \varkappa \ln \left(\frac{\sin(\varkappa+1)\phi}{\sin \varkappa\phi} \right) + \ln \left(\frac{\sin(\varkappa+1)\phi}{\sin \phi} \right) &= 2 \ln(2 \cos \phi), \quad \gamma_0 = -\ln 2, \\ \rho(\gamma) &= \begin{cases} \frac{2}{\pi} \arcsin\left(\frac{e^{-\gamma}}{2}\right), & \gamma > -\ln 2, \\ 1, & \gamma < -\ln 2. \end{cases} \end{aligned}$$

Since $\frac{2}{\pi} \arcsin\left(\frac{e^{-\gamma}}{2}\right) = \frac{1}{\pi} \arccos\left(1 - \frac{e^{-2\gamma}}{2}\right)$, in this case the global limit density coincides (up to rescaling of γ by 2) with that for the largest section of random plane partitions with uniform measure, see §3.1.10 of [OR1] with $\tau = 0$.

4. Equal time values of the limit correlation kernel equal

$$\mathcal{K}_{\tau,\tau}(x-y) = \frac{1}{2\pi i} \int_{\zeta(\phi)}^{\overline{\zeta(\phi)}} \frac{d\zeta}{\zeta^{x-y+1}} = \left(\frac{\sin \varkappa\phi}{\sin \phi} \right)^{x-y} \frac{\sin((\pi - (1+\varkappa)\phi)(x-y))}{\pi(x-y)}.$$

The prefactor $\left(\frac{\sin \varkappa\phi}{\sin \phi} \right)^{x-y}$ can be ignored as it cancels out in the determinantal expression for the correlation functions, and we obtain the discrete sine kernel. Hence, the full kernel $\mathcal{K}_{\sigma,\tau}(x-y)$ can be viewed as an extension of the discrete sine kernel. Let us show that this extension belongs to the family of extensions constructed in Theorem 4.4 above. For $\sigma \leq \tau$ we have

$$\begin{aligned} \mathcal{K}_{\sigma,\tau}(x-y) &= (-1)^{A(\sigma,\tau]} \frac{1}{2\pi i} \int_{\zeta(\phi)}^{\overline{\zeta(\phi)}} \frac{(1-\zeta)^{\tau-\sigma} d\zeta}{\zeta^{x-y+1+A(\sigma,\tau]}} \\ &= (-1)^{A(\sigma,\tau]} \left(\frac{\sin \varkappa\phi}{\sin \phi} \right)^{x-y+A(\sigma,\tau]} \frac{1}{2\pi i} \int_{e^{i((1+\varkappa)\phi-\pi)}}^{e^{i(\pi-(1+\varkappa)\phi)}} \frac{\left(1 - \frac{\sin \phi}{\sin \varkappa\phi} \zeta\right)^{\tau-\sigma} d\zeta}{\zeta^{x-y+1+A(\sigma,\tau]}} \end{aligned}$$

while for $\sigma > \tau$

$$\mathcal{K}_{\sigma,\tau}(x-y) = -(-1)^{A(\tau,\sigma]} \left(\frac{\sin \varkappa\phi}{\sin \phi} \right)^{x-y-A(\tau,\sigma]} \frac{1}{2\pi i} \int_{e^{i(\pi-(1+\varkappa)\phi)}}^{e^{i((1+\varkappa)\phi-\pi)}} \frac{\left(1 - \frac{\sin \phi}{\sin \varkappa\phi} \zeta\right)^{\sigma-\tau} d\zeta}{\zeta^{x-y+1-A(\tau,\sigma]}}.$$

Once again, the prefactors can be ignored, and after shifting the space variable x at time τ by $x \mapsto x + A(T, \tau]$ for a fixed T and all τ , we see that the correlation functions are determinants of the kernel

$$\begin{cases} \frac{1}{2\pi i} \int \frac{e^{i(\pi - (1+\kappa)\phi)} (1 - \frac{\sin \phi}{\sin \kappa \phi} \zeta)^{\tau - \sigma} d\zeta}{e^{i((1+\kappa)\phi - \pi)} \zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int \frac{e^{i((1+\kappa)\phi - \pi)} (1 - \frac{\sin \phi}{\sin \kappa \phi} \zeta)^{\sigma - \tau} d\zeta}{e^{i(\pi - (1+\kappa)\phi)} \zeta^{x-y+1}}, & \sigma > \tau, \end{cases}$$

where the integrals are taken over positively oriented arches of the unit circle.

For $\kappa = 1$ this is exactly the incomplete beta kernel arising in the bulk limit of uniform measures on plane partitions, see [OR1].

If $\kappa > 1$ then $\frac{\sin \phi}{\sin \kappa \phi} < 1$ and the kernel coincide with one of the stationary kernels afforded by Theorem 4.4; one has to take $H(a[k]; u) \equiv 1$, $H(b[k]; u) = (1 - \frac{\sin \phi}{\sin \kappa \phi} u)^{-1}$ for all $k \in \mathbb{Z}$.

Finally, if $\kappa < 1$ then $\frac{\sin \phi}{\sin \kappa \phi} > 1$, and one more shift of the space variable x at time τ by $x \mapsto x - T + \tau$ (or, equivalently, the shift $x \mapsto x - B(T, \tau]$ of the initial space variable x at time τ) for a fixed T and all τ , brings our kernel, up to irrelevant prefactors, to the kernel of Theorem 4.4 with $H(a[k]; u) = (1 - \frac{\sin \kappa \phi}{\sin \phi} u)^{-1}$ and $H(b[k]; u) \equiv 1$ for all $k \in \mathbb{Z}$.

Observe that all these kernels are invariant with respect to time shifts. Thus, the bulk behavior of our cylindric partitions with slowly growing period N is rotationally invariant and independent of the initial profile except for its slope, after we perform the above shifting of the space variables. However, these shifts do depend on the profile.

Proof of Theorems 6.2 and 6.3. As in the proof of Theorem 3.1, let us introduce a new integration variable $\xi = \zeta \eta$ and rewrite the kernel from Proposition 5.2 in the form

$$K(\sigma, x; \tau, y) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)^2} \oint_{\xi} \oint_{\zeta} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})} \frac{\theta_3(z \xi; t)}{\theta_3(-\xi t^{-\frac{1}{2}}; t)} \frac{d\zeta d\xi}{\zeta^{x-y+1} \xi^{y+\frac{1}{2}}},$$

where, by Remark 5.4,

$$\begin{aligned} \frac{F(\sigma, \zeta)}{F(\tau, \zeta \xi^{-1})} &= \frac{\prod_{k \in \overline{1, \tau}: B[k]=1} (1 - s^{-k} \zeta / \xi)}{\prod_{k \in \overline{1, \sigma}: B[k]=1} (1 - s^{-k} \zeta)} \frac{\prod_{k \in \overline{\sigma+1, N}: A[k]=1} (1 - s^k / \zeta)}{\prod_{k \in \overline{\tau+1, N}: A[k]=1} (1 - s^k \xi / \zeta)} \\ &\quad \times \prod_{k \in \overline{1, N}: B[k]=1} \frac{(s^{-k} t \zeta / \xi; t)_{\infty}}{(s^{-k} t \zeta; t)_{\infty}} \prod_{k \in \overline{1, N}: A[k]=1} \frac{(s^k t / \zeta; t)_{\infty}}{(s^k t \xi / \zeta; t)_{\infty}}. \end{aligned}$$

Here, according to Proposition 5.2, the integrals are taken over circles centered at the origin with radii satisfying $s^{\sigma+1} < |\zeta| < s^{\sigma}$ and $s^{\tau+1} < |\zeta|/|\xi| < s^{\tau}$ plus the additional condition $|\xi| > 1$ if $\sigma = \tau$.

The proof proceeds along the same lines as that of Theorem 3.1. The only major obstacle is the unboundedness of the ratio $F(\sigma, \zeta)/F(\tau, \zeta \xi^{-1})$ on the integration contours. Recall that in Theorem 3.1 we imposed the condition $A_k = \overline{B_k}$ which

ensured the needed boundedness when $|\zeta|$ and $|\xi|$ were close to 1. Here we only have such “self-adjointness” in the case $\varkappa = 1$; for general $\varkappa > 0$ we need to deform the ζ -integration contour.

Clearly, the first factor in the formula for $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$ above remains bounded as $s \rightarrow 1$ as long as the factors in the denominator do not approach zero. Furthermore, as $\xi \rightarrow 1$ and $s \rightarrow 1$, it tends to $(1 - \zeta)^{B(\sigma, \tau]}(1 - \zeta^{-1})^{A(\sigma, \tau]}$ for $\sigma \leq \tau$, and to $(1 - \zeta)^{-B(\tau, \sigma]}(1 - \zeta^{-1})^{-A(\tau, \sigma]}$ for $\sigma > \tau$. As we will see, only integration over infinitesimally close to 1 ξ 's yields a nonzero contribution, and this asymptotics gives the corresponding factors in the final formula for the correlation functions.

The second factor needs more attention. Recall that the dilogarithm function is defined in the unit disc by the power series

$$\operatorname{dilog}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Its analytic continuation is provided by the integral representation

$$\operatorname{dilog}(z) = - \int_0^z \frac{\ln(1-x) dx}{x},$$

which shows that $z = 1$ is the branching point of this function. We will consider $\operatorname{dilog}(z)$ as the holomorphic function on $\mathbb{C} \setminus (1, +\infty)$ defined by the integral above with the principal branch of $\ln(1-x)$. Note that the jump of $\operatorname{dilog}(z)$ across $(1, +\infty)$ is purely imaginary, which means that $\Re \operatorname{dilog}(z)$ is a continuous function on \mathbb{C} .

Our interest in $\operatorname{dilog}(\cdot)$ is explained by the fact that

$$\ln(x; t)_{\infty} = \sum_{n=0}^{\infty} \ln(1 - xt^n) = \frac{1}{1-t} \sum_{n=0}^{\infty} \frac{(xt^n - xt^{n+1}) \ln(1 - xt^n)}{xt^n}$$

is, up to the factor $(1-t)^{-1}$, a Riemannian sum for the integral representation of $\operatorname{dilog}(x)$. Thus, using the mean value theorem to estimate the remainder, for $x \in \mathbb{C} \setminus [1, +\infty)$ we obtain

$$\ln(x; t)_{\infty} = -(1-t)^{-1} \operatorname{dilog}(x) + O(\operatorname{dist}(1, \{rx \mid 0 \leq r \leq 1\})^{-1}), \quad t \rightarrow 1,$$

and the estimate is uniform on compact sets. (The remainder may become large as x approaches the cut $[1, +\infty)$.)

This asymptotic relation shows that the second factor in the formula for the ratio $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$, as $t \rightarrow 1$, is approximated by

$$\exp \left(- \frac{l(\operatorname{dilog}(\zeta/\xi) - \operatorname{dilog}(\zeta)) + d(\operatorname{dilog}(1/\zeta) - \operatorname{dilog}(\xi/\zeta))}{1-t} \right).$$

Note also that, since $\operatorname{dilog}'(x) = -\ln(1-x)/x$, as $\xi \rightarrow 1$ we have

$$l(\operatorname{dilog}(\zeta/\xi) - \operatorname{dilog}(\zeta)) + d(\operatorname{dilog}(1/\zeta) - \operatorname{dilog}(\xi/\zeta)) \sim \ln \xi (l \ln(1-\zeta) + d \ln(1-\zeta^{-1})),$$

which coincides, up to multiplication by $d \ln \xi$, with function $f_{\varkappa}(\zeta)$ introduced in Lemma 6.1 above.

Lemma 6.4. *For any $\varkappa > 0$ and $\xi \neq 1$ on the unit circle, set*

$$f_{\varkappa,\xi}(\zeta) = \varkappa(\operatorname{dilog}(\zeta/\xi) - \operatorname{dilog}(\zeta)) + \operatorname{dilog}(1/\zeta) - \operatorname{dilog}(\xi/\zeta).$$

Then $f_{\varkappa,\xi}$ is a holomorphic function on $\mathbb{C} \setminus \{\mathbb{R}_+ \cup \xi\mathbb{R}_+\}$, its real part is continuous on \mathbb{C} , and

$$\{\zeta \in \mathbb{C} \mid \Re f_{\varkappa,\xi}(\zeta) = 0\} = \sqrt{\xi} \cdot \mathbb{R} \cup \Gamma_{\varkappa,\xi},$$

where $\Gamma_{\varkappa,\xi}$ is a piecewise smooth closed curve which encloses the origin and intersects each ray $e^{i\varphi}\mathbb{R}_+ \not\subseteq \sqrt{\xi} \cdot \mathbb{R}$ at a single point. Further, $\Re f_{\varkappa,\xi}(\zeta) < 0$ if and only if ζ is inside $\Gamma_{\varkappa,\xi}$ and to the left of the line $\sqrt{\xi} \cdot \mathbb{R}$, or ζ is outside $\Gamma_{\varkappa,\xi}$ and to the right of the line $\sqrt{\xi} \cdot \mathbb{R}$.

The curve $\Gamma_{\varkappa_1,\xi}$ lies inside $\Gamma_{\varkappa_2,\xi}$ if $\varkappa_1 > \varkappa_2$; the curve $\Gamma_{1,\xi}$ is the unit circle. Also, as $\xi \rightarrow 1$, we have $\Gamma_{\varkappa,\xi} \rightarrow \Gamma_{\varkappa}$ in the sense that the intersection points of $\Gamma_{\varkappa,\xi}$ with rays $e^{i\varphi}\mathbb{R}_+$ converge to the corresponding intersection points of Γ_{\varkappa} .

Proof. The fact that $f_{\varkappa,\xi}$ takes real values on $\sqrt{\xi} \cdot \mathbb{R}$ follows from the relation $\operatorname{dilog}(x) = \operatorname{dilog}(\bar{x})$. On the other hand, on any ray we compute

$$\Re \frac{df_{\varkappa,\xi}(re^{i\varphi})}{dr} = \frac{1+\varkappa}{r} \ln \left| \frac{1-re^{i\varphi}}{\xi-re^{i\varphi}} \right|.$$

This expression is negative if ζ is to the right of the line $\sqrt{\xi} \cdot \mathbb{R}$, and it is positive if ζ is to the left of the line $\sqrt{\xi} \cdot \mathbb{R}$.

Using the asymptotic relation

$$\lim_{r \rightarrow +\infty} \Re(\operatorname{dilog}(-re^{i\psi_1}) - \operatorname{dilog}(-re^{i\psi_2})) = \frac{1}{2}(\psi_1^2 - \psi_2^2), \quad -\pi \leq \psi_1, \psi_2 \leq \pi,$$

we see that the limits of $f_{\varkappa,\xi}(re^{i\varphi})$ as $r \rightarrow 0+$ and $r \rightarrow +\infty$ exist, and if $e^{i\varphi}\mathbb{R}_+ \not\subseteq \sqrt{\xi} \cdot \mathbb{R}$ then they are nonzero and have different signs. Since the real part of the derivative along any such ray is sign definite, this implies the existence and uniqueness of the needed intersection points as well as the inequalities on $\Re f_{\varkappa,\xi}(\zeta)$.

Similar arguments show that $\frac{df_{\varkappa,\xi}(\zeta)}{d\zeta}$ vanishes at exactly two points located in $\sqrt{\xi} \cdot \mathbb{R}_+$ and $\sqrt{\xi} \cdot \mathbb{R}_-$. These are the points where $\Gamma_{\varkappa,\xi}$ intersects the critical line $\sqrt{\xi} \cdot \mathbb{R}$. Thus, the curve is closed, and a standard implicit function theorem argument implies the smoothness.

Since the limit values $\lim_{r \rightarrow 0+} f_{\varkappa,\xi}(re^{i\varphi})$ do not depend on \varkappa , while the absolute value of the derivative $|\Re \frac{df_{\varkappa,\xi}(re^{i\varphi})}{dr}|$ is an increasing (and linear) function of \varkappa , we see that the larger \varkappa the sooner $f_{\varkappa,\xi}(re^{i\varphi})$ reaches 0 as r increases. This implies the inclusion property of the curves $\Gamma_{\varkappa,\xi}$. Finally, the convergence of $\Gamma_{\varkappa,\xi}$ to Γ_{\varkappa} follows from the asymptotic relation $f_{\varkappa,\xi}(\zeta) \sim \ln \xi f_{\varkappa}(\zeta)$ as $\xi \rightarrow 1$, Lemma 6.1, and the fact that $\ln \xi$ is purely imaginary as $|\xi| = 1$. \square

From this moment the proof very much resembles that of Theorem 3.1. For ξ bounded away from the point 1 we use Lemma 6.4 to deform the ζ -integration contour to the one where the ratio $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$ remains bounded as $t \rightarrow 1$. The result of Proposition 3.2(i) then implies that the integral over such ξ tends to zero. On the other hand, for ξ close to 1 we deform the ζ -contour to Γ_{\varkappa} and use

Proposition 3.2(i) together with the asymptotics of $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$ to see that (cf. the formula before Lemma 3.3)

$$K(\sigma, x; \tau, y) \sim \frac{1}{2(i\pi)^2} \int_{\Gamma_\infty} \frac{\prod_{k \in \overline{1, \tau}: B[k]=1} (1 - \zeta) \prod_{k \in \overline{\sigma+1, N}: A[k]=1} (1 - 1/\zeta)}{\prod_{k \in \overline{1, \sigma}: B[k]=1} (1 - \zeta) \prod_{k \in \overline{\tau+1, N}: A[k]=1} (1 - 1/\zeta)} \\ \times \int_{-\infty \pm i\varepsilon}^{+\infty \pm i\varepsilon} \frac{e^{\frac{i u}{\pi} (\ln z - \gamma - l \ln(1 - \zeta) - d \ln(1 - \zeta^{-1}))}}{e^u - e^{-u}} \frac{d u d \zeta}{\zeta^{x-y+1}},$$

as $t \rightarrow 1$, $\varepsilon > 0$ is small enough, the sign in $\pm i\varepsilon$ is “+” for $\sigma \leq \tau$ and “−” for $\sigma > \tau$, and γ is replaced by γN in the case of Theorem 6.3. The integration variable u is related to ξ via $u = -i\pi r^{-1} \ln \xi$.

The integral over u is evaluated by Lemma 3.3 to give

$$\frac{\pm i\pi}{1 + (z^{-1} e^\gamma (1 - \zeta)^l (1 - \zeta^{-1})^d)^{\pm 1}}.$$

Finally, similarly to the proof of Theorem 3.1, the integration over z (see the formula of Proposition 5.2) is shown to be asymptotically equivalent to substituting $z = 1$.

In the case of Theorem 6.3 the argument is exactly the same except for the fact that $e^\gamma (1 - \zeta)^l (1 - \zeta^{-1})^d$ converges to either 0 or $+\infty$, see Comment 1 to Theorem 6.3. However, Lemma 3.3 can still be applied because its estimates remain uniform as long as (in the notation of Lemma 3.3) $\varepsilon \Re a$ is uniformly bounded and $|\Im a|$ is bounded away from 1. Since ε can be made arbitrarily small, $\Re a$ is allowed to converge to $\pm\infty$.

We skipped the discussion of several technical issues here. One should make sure that the deformation of ζ -contours happens in such a way that the poles of the ratio $F(\sigma, \zeta)/F(\tau, \zeta\xi^{-1})$ are not passed in the process; the remainder in the approximation of $\ln(x, t)_\infty$ by $\text{dilog}(x)$ may grow as x approaches cut $x \in (1, +\infty)$, and one needs to control the growth of the derivative of the remainder term in order to apply Lemma 3.3. All these issues can be resolved. However, the arguments are rather tedious although fairly straightforward, and we omit them. \square

7. BULK OF LARGE CYLINDRIC PARTITIONS WITH PERIOD OF INTERMEDIATE GROWTH

In the previous section we have seen that if the period N is small comparing to $|\ln s|^{-1}$ then the global density function for the uniform measure on cylindric partitions with a fixed profile, when the partitions are scaled by $|\ln s|$, converges to a rotationally invariant limit which depends only on the average slope of the profile.

In this section we consider the case when $t = s^N$ has a nontrivial limit strictly between 0 and 1. Equivalently, $N \ln s^{-1} \rightarrow \ln t^{-1} \in (0, +\infty)$. This situation is much more complex because, as we will see, the global density is no longer rotationally invariant.

It is worth noting that if N grows fast enough (so that $t \rightarrow 0$) then the random cylindric partitions split into disjoint independent random plane partitions located at the “corners” of the profile provided that these corners are deep enough.

We will consider in detail two cases: the most rotationally symmetric one with $\{A[k]\}$ and $\{B[k]\}$ being periodic of finite period (in the notation of §5 this means that μ is the staircase-like partition), and the most rotationally asymmetric one

with $\{A[k]\}$ and $B[k]\}$ consisting of two blocks of zeroes and ones (this means that μ is either empty or it is the rectangle $d \times l$).

In both cases we will assume that the average slope \varkappa is equal to 1. This will allow us to obtain asymptotic results without substantially deforming the contours in our integral representation for the correlation kernel. We hope to consider more general cases in a subsequent publication.

Proposition 7.1. *Consider the uniform measure on cylindric partitions with a given profile $\{A[k], B[k]\}_{k=1}^N$ and assume that the sequences $\{A[k]\}$ and $\{B[k]\}$ are periodic with an even period M and $A(0, M) = B(0, M) = M/2$. Then as*

$$s \rightarrow 1-, \quad N \rightarrow \infty, \quad s^N \rightarrow t \in (0, 1),$$

the limit of the correlation functions of this measure is described by Theorem 6.3 above with $\varkappa = 1$ (see also Comment 3 after Theorem 6.3).

Proof. For the sake of convenience let us consider the case when the large period N is a multiple of the small period M ; $N/M \in \mathbb{Z}$. This assumption is by no means necessary and can be easily removed.

Also, using the rotational invariance, we may assume without loss of generality that all the time moments $\tau_1(s), \dots, \tau_n(s)$ from the statement of Theorem 6.3 remain positive and uniformly bounded as $s \rightarrow 1$.

As in the proof of Theorems 6.2 and 6.3, we start with the integral representation for the correlation kernel $K(\sigma, x; \tau, y)$ given in Proposition 5.2. We have

$$\begin{aligned} \frac{F(\sigma, \zeta)}{F(\tau, \eta^{-1})} &= \frac{\prod_{k \in \overline{1, \tau}: B[k]=1} (1 - s^{-k}/\eta)}{\prod_{k \in \overline{1, \sigma}: B[k]=1} (1 - s^{-k}\zeta)} \frac{\prod_{k \in \overline{1, \tau}: A[k]=1} (1 - s^k\eta)}{\prod_{k \in \overline{1, \sigma}: A[k]=1} (1 - s^k/\zeta)} \\ &\quad \times \prod_{k \in \overline{1, N}: B[k]=1} \frac{(s^{N-k}/\eta; t)_\infty}{(s^{N-k}\zeta; t)_\infty} \prod_{k \in \overline{1, N}: A[k]=1} \frac{(s^k/\zeta; t)_\infty}{(s^k\eta; t)_\infty}. \end{aligned}$$

Using the periodicity assumption, we can rewrite the second factor as

$$\prod_{k \in \overline{1, M}: B[k]=1} \frac{(s^{M-k}/\eta; s^M)_\infty}{(s^{M-k}\zeta; s^M)_\infty} \prod_{k \in \overline{1, M}: A[k]=1} \frac{(s^k/\zeta; s^M)_\infty}{(s^k\eta; s^M)_\infty}$$

(here the base t of all products $(x; t)_\infty$ changed from t to s^M).

Starting from this moment the proof almost literally repeats the proof of Theorem 2 in [OR1], see §§3.1.3-3.1.9. Approximating $(x; s^M)$ by the dilogarithm function (see the previous section) and keeping in mind that $A(0, M) = B(0, M) = M/2$, we obtain that as $s \rightarrow 1$ the second factor in the formula for the ratio $F(\sigma, \zeta)/F(\tau, \eta)$ above is approximated by

$$\exp \left(\frac{M(\text{dilog}(\zeta) - \text{dilog}(1/\zeta) + \text{dilog}(\eta) - \text{dilog}(1/\eta))}{2(1 - s^M)} \right).$$

Note that $1 - s^M$ can be replaced by $M|\ln s|$ as $s \rightarrow 1$.

The only other two factors in the integral representation of $K(\sigma, x; \tau, y)$ which may not have a finite limit as $s \rightarrow 1$ are $\zeta^{-x-\frac{1}{2}}\eta^{-y-\frac{1}{2}}$. Since $x, y \sim \gamma/|\ln s|$, we can include these factors into the exponential above to obtain

$$\exp \left(\frac{\text{dilog}(\zeta) - \text{dilog}(1/\zeta) - 2\gamma \ln \zeta + \text{dilog}(\eta) - \text{dilog}(1/\eta) - 2\gamma \ln \eta}{2|\ln s|} \right).$$

Following §3.1 of [OR1] we see that the function

$$\Re(\operatorname{dilog}(\zeta) - \operatorname{dilog}(1/\zeta) - 2\gamma \ln \zeta),$$

which clearly vanishes on the unit circle $|\zeta| = 1$, is negative slightly inside the unit circle if $-\frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2}) \leq \arg \zeta \leq \frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2})$ and slightly outside the unit circle if $|\arg \zeta| > \frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2})$; the points $\arg \zeta = \pm \frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2})$ with $|\zeta| = 1$ are the critical points of the function $\operatorname{dilog}(1/\zeta) - \operatorname{dilog}(\zeta) - 2\gamma \ln \zeta$.

We would like to deform both ζ - and η -integration contours to put them in the domain where the exponential above converges to zero, i.e. where

$$\Re(\operatorname{dilog}(\zeta) - \operatorname{dilog}(1/\zeta) - 2\gamma \ln \zeta) < 0, \quad \Re(\operatorname{dilog}(\eta) - \operatorname{dilog}(1/\eta) - 2\gamma \ln \eta) < 0$$

(this corresponds to using the contour $\gamma_<$ from [OR1]). However, there is an obstacle: the function $\theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)$, which enters the denominator of the integrand in the formula for $K(\sigma, x; \tau, y)$, vanishes when $\zeta\eta = 1$. Thus, while deforming the contours we have to add the residues corresponding to the poles $\zeta\eta = 1$.

Consider the case $\sigma \leq \tau$ first. Then Proposition 5.2 states that the ζ - and η -contours are such that η -contour contains the $1/\zeta$ -contour. Since our desired contours go inside the unit circle when the arguments of ζ and η are inside the interval $(-\frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2}), \frac{1}{2} \arcsin(\frac{e^{-\gamma}}{2}))$, this is where the residues have to be added. For $\gamma < -\ln 2$ the residues have to be taken along the whole integration contour.

Observe that

$$\begin{aligned} -\operatorname{Res}_{\eta=1/\zeta} & \frac{\prod_{n \geq 1} (1-t^n)^3 \theta_3(z\zeta\eta; t)}{\theta_3(z; t) \theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)} \frac{F(\sigma, \zeta)}{F(\tau, \eta)} \frac{1}{\zeta^{x+\frac{1}{2}} \eta^{y+\frac{1}{2}}} \\ &= \frac{\prod_{k \in \overline{\sigma+1}, \tau: B[k]=1} (1-s^{-k}\zeta) \prod_{k \in \overline{\sigma+1}, \tau: A[k]=1} (1-s^k/\zeta)}{\zeta^{x-y+1}}, \end{aligned}$$

which asymptotically equals $(1-\zeta)^{B(\sigma, \tau]} (1-\zeta^{-1})^{A(\sigma, \tau]} / \zeta^{x-y+1}$ as $s \rightarrow 1$. (Recall that we assumed in the beginning of the proof that the time moments σ, τ remain bounded as $s \rightarrow 1$.) Thus, the residue contribution is asymptotically equal to

$$\frac{1}{2\pi i} \int_{e^{-\frac{i}{2} \arcsin(\frac{e^{-\gamma}}{2})}}^{e^{\frac{i}{2} \arcsin(\frac{e^{-\gamma}}{2})}} (1-\zeta)^{B(\sigma, \tau]} (1-\zeta^{-1})^{A(\sigma, \tau]} \frac{d\zeta}{\zeta^{x-y+1}}.$$

On the other hand, the integral when ζ and η range over the desired contour converges to zero as $s \rightarrow 1$, see §3.1.1 of [OR1] for additional explanations.

The arguments for $\sigma > \tau$ are very similar.

Note that all the estimates are uniform in the parameter z , $|z| = 1$, and the final asymptotic expressions are independent of z . This means that the integration over z in the formula of Proposition 5.2 in the limit $s \rightarrow 1$ can be simply removed. \square

Now let us consider the non-symmetric case. Take $d = l = N/2$ and

$$\{A[k]\}_{k=1}^N = (\underbrace{1, 1, \dots, 1}_{N/2}, \underbrace{0, \dots, 0}_{N/2}), \quad \{B[k]\}_{k=1}^N = (\underbrace{0, 0, \dots, 0}_{N/2}, \underbrace{1, \dots, 1}_{N/2})$$

(we are assuming that N is even). This corresponds to the square partition $\mu = (\frac{N}{2})^{\frac{N}{2}}$, and the random sequences of ordinary partitions of the form

$$\lambda^{(1)} \succ \lambda^{(2)} \succ \dots \succ \lambda^{(\frac{N}{2}+1)} \prec \lambda^{(\frac{N}{2}+2)} \prec \dots \prec \lambda^{(N)} \prec \lambda^{(N+1)} = \lambda^{(1)}.$$

We will describe the limit of the correlation functions near the largest partition $\lambda^{(1)}$ and near the smallest partition $\lambda^{(\frac{N}{2}+1)}$. (Recall that the time variable of the periodic Schur process or, equivalently, the index of the Young diagrams above, may be viewed as an element of $\mathbb{Z}/N\mathbb{Z}$.)

Theorem 7.2. *In the limit*

$$s \rightarrow 1-, \quad N \rightarrow \infty, \quad s^N \rightarrow t \in (0, 1)$$

the correlation functions of the uniform measure on cylindric partitions with the profile as above have limits in the following sense: Choose $x_1(s), \dots, x_n(s) \in \mathbb{Z} + \frac{1}{2}$ such that as $s \rightarrow 1-$, $|\ln s| \cdot x_k(s) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j = x_i(s) - x_j(s)$ remain constant.

(i) Choose the time moments $\tau_1(s), \dots, \tau_n(s) \in \{-\frac{N}{2}, \dots, \frac{N}{2}\}$ such that as $s \rightarrow 1$ the absolute values $|\tau_i(s)|$ remain uniformly bounded. Then

$$\rho_n(\tau_1(s), x_1(s); \dots, \tau_n(s), x_n(s)) = \det[K_{\tau_i(s), \tau_j(s)}^{(\gamma)}(x_i - x_j)]_{i,j=1}^n + o(1),$$

where the correlation kernel has the following form:

$$\mathcal{K}_{\tau, \tau}^{(\gamma)}(x - y) = \delta(x - y) \quad \text{if } \gamma \leq \gamma_1^{\min}(t) := 2 \ln \frac{(-\sqrt{t}; t)_\infty}{(-1; t)_\infty},$$

and for $\gamma > \gamma_1^{\min}(t)$

$$\mathcal{K}_{\sigma, \tau}^{(\gamma)}(x - y) = \begin{cases} \frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} (1 - \zeta)^{B(\sigma, \tau)} (1 - \zeta^{-1})^{A(\sigma, \tau]} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma \leq \tau, \\ -\frac{1}{2\pi i} \int_{e^{-ic}}^{e^{ic}} (1 - \zeta)^{-B(\tau, \sigma]} (1 - \zeta^{-1})^{-A(\tau, \sigma]} \frac{d\zeta}{\zeta^{x-y+1}}, & \sigma > \tau, \end{cases}$$

where the constant $c = c(\gamma) \in (0, \pi)$ from the limits of integration is uniquely determined by the relation

$$\gamma = 2 \ln \left| \frac{(e^{ic}\sqrt{t}; t)_\infty}{(e^{ic}; t)_\infty} \right|.$$

(ii) Choose the time moments $\tau_1(s), \dots, \tau_n(s) \in \{1, \dots, N\}$ such that as $s \rightarrow 1$ the absolute values $|\tau_i(s) - \frac{N}{2}|$ remain uniformly bounded. Then the asymptotics of the correlation functions has the determinantal form as in (i) above with the correlation kernel given by

$$K_{\tau, \tau}^{(\gamma)}(x - y) = \begin{cases} \delta(x - y), & \text{if } \gamma \leq \gamma_2^{\min}(t) := 2 \ln \frac{(-t; t)_\infty}{(-\sqrt{t}; t)_\infty}, \\ 0, & \text{if } \gamma \geq \gamma_2^{\max}(t) := 2 \ln \frac{(t; t)_\infty}{(\sqrt{t}; t)_\infty}, \end{cases}$$

and for $\gamma \in (\gamma_2^{\min}(t), \gamma_2^{\max}(t))$ it is given by the formula of (i) above, where the constant $c = c(\gamma) \in (0, \pi)$ from the limits of integration is uniquely determined by the relation

$$\gamma = 2 \ln \left| \frac{(e^{ic}t; t)_\infty}{(e^{ic}\sqrt{t}; t)_\infty} \right|.$$

Comments. 1. For any $t \in (0, 1)$ the functions

$$\gamma_1(t, c) = 2 \ln \left| \frac{(e^{ic}\sqrt{t}; t)_\infty}{(e^{ic}; t)_\infty} \right|, \quad \gamma_2(t, c) = 2 \ln \left| \frac{(e^{ic}t; t)_\infty}{(e^{ic}\sqrt{t}; t)_\infty} \right|$$

as functions of c are strictly decreasing on $(0, \pi)$, and

$$\begin{aligned} \lim_{c \rightarrow 0} \gamma_1(t, c) &= +\infty, & \lim_{c \rightarrow \pi} \gamma_1(t, c) &= \gamma_1^{\min}(t), \\ \lim_{c \rightarrow 0} \gamma_2(t, c) &= \gamma_2^{\max}(t), & \lim_{c \rightarrow \pi} \gamma_2(t, c) &= \gamma_2^{\min}(t). \end{aligned}$$

Indeed, this follows from the fact that for any $\alpha, \beta \in (0, 1)$ and $c \in (0, \pi)$, we have

$$\frac{d}{dc} \left| \frac{1 - \alpha\beta e^{ic}}{1 - \alpha e^{ic}} \right|^2 = - \frac{2\alpha(1 - \beta)(1 - \alpha^2\beta) \sin(c)}{|1 - \alpha e^{ic}|^4} < 0.$$

The decay of $\gamma_1(t, c)$ and $\gamma_2(t, c)$ guarantees the existence of the unique $c \in (0, \pi)$ satisfying the needed relation.

2. The global density function in both (i) and (ii) is given by

$$\rho(\gamma) = \begin{cases} 1, & \gamma \leq \gamma_{1,2}^{\min}(t), \\ \frac{1}{\pi} c(\gamma), & \gamma_{1,2}^{\min}(t) \leq \gamma \leq \gamma_{1,2}^{\max}(t), \\ 0, & \gamma \geq \gamma_{1,2}^{\max}(t), \end{cases}$$

where $\gamma_{1,2}^{\max}(t) = +\infty$. In particular, in (i) we see that the random Young diagrams have only the lower edge, while in (ii) there are both the lower and the upper edges.

3. The functions $\gamma_{1,2}(t, c)$ can be written in terms of the Jacobi elliptic sine function:

$$\begin{aligned} \left| \frac{(e^{ic}\sqrt{t}; t)_\infty}{(e^{ic}; t)_\infty} \right|^2 &= \frac{t^{\frac{1}{8}}}{2\sqrt{k}} \frac{1}{\operatorname{sn}\left(\frac{Kc}{\pi}, k\right) \sin\left(\frac{c}{2}\right)}, & \left| \frac{(e^{ic}t; t)_\infty}{(e^{ic}\sqrt{t}; t)_\infty} \right|^2 &= \frac{\sqrt{k}}{2t^{\frac{1}{8}}} \frac{\operatorname{sn}\left(\frac{Kc}{\pi}, k\right)}{\sin\left(\frac{c}{2}\right)}, \\ k &= \frac{\theta_2^2(0; t)}{\theta_3^2(0; t)}, & K &= \frac{\pi}{2} \theta_3^2(0; t). \end{aligned}$$

Note that as $t \rightarrow 1$, the elliptic sine tends to 1, and the formulas of both (i) and (ii) degenerate to those of Comment 3 after Theorem 6.3. This agrees with the fact that if N is growing slowly then the global limit density function is rotationally invariant.

On the other hand, as $t \rightarrow 0$ in the setting of (i) we have

$$e^\gamma = \frac{|(e^{ic(\gamma)}\sqrt{t}; t)_\infty|^2}{|(e^{ic(\gamma)}; t)_\infty|^2} \rightarrow \frac{1}{|1 - e^{ic(\gamma)}|^2} = \frac{1}{2(1 - \cos c(\gamma))} \quad \text{or} \quad c(\gamma) \rightarrow \arccos\left(1 - \frac{e^{-\gamma}}{2}\right)$$

which is exactly the behavior of the largest section of the random plane partitions, cf. §3.1.10 of [OR1] with $\tau = 0$. This agrees with the fact that if N grows too fast then the random cylindric partitions split into disjoint independent plane partitions located in deep corners of the profile (in this case we have only one such corner).

Proof. The proof is very similar to that of Proposition 7.1 and [OR1, Theorem 2]. The only difference from the arguments used for Proposition 7.1 is in the specifics of the functions $F(\tau, \zeta)$. Using Remark 5.4, for bounded $\tau > 0$ we have

$$F(\tau, \zeta) = \prod_{k=1}^{\tau} (1 - s^k / \zeta)^{-1} \frac{\prod_{k=1}^{N/2} (s^k / \zeta; t)_{\infty}}{\prod_{k=0}^{N/2-1} (s^k \zeta; t)_{\infty}}.$$

The first factor has the finite limit $(1 - \zeta^{-1})^{-\tau}$, while for the second factor we have

$$\begin{aligned} \prod_{k=0}^{N/2-1} (s^k x; t)_{\infty} &= \frac{(x; s)_{\infty} (xt; s)_{\infty} (xt^2; s)_{\infty} \cdots}{(x\sqrt{t}; s)_{\infty} (xt\sqrt{t}; s)_{\infty} (xt^2\sqrt{t}; s)_{\infty} \cdots} \\ &\sim \exp \frac{1}{|\ln s|} \sum_{m=0}^{\infty} (\text{dilog}(xt^m \sqrt{t}) - \text{dilog}(xt^m)). \end{aligned}$$

Thus, in $F(\tau, \zeta)$ we have the factor

$$\exp \frac{1}{|\ln s|} \sum_{m=0}^{\infty} (\text{dilog}(\zeta t^m) - \text{dilog}(\zeta t^m \sqrt{t}) - \text{dilog}(t^m / \zeta) + \text{dilog}(t^m \sqrt{t} / \zeta))$$

times the part which has a finite limit as $s \rightarrow 1$.

The real part of the above sum is identically equal to zero on the unit circle $|\zeta| = 1$. Furthermore, the derivative of this sum with respect to ζ multiplied by ζ equals (for $|\zeta| = 1$)

$$-2 \ln \prod_{m \geq 0} \left| \frac{1 - \zeta t^m}{1 - \zeta t^m \sqrt{t}} \right| = 2 \ln \left| \frac{(\zeta \sqrt{t}; t)_{\infty}}{(\zeta; t)_{\infty}} \right|.$$

From Comment 1 above we know that for any $\gamma \in (\gamma_1^{\min}, +\infty)$ this expression is strictly greater than γ on the arch $\arg \zeta \in (-c(\gamma), c(\gamma))$, where $c = c(\gamma)$ satisfies the relation in the hypothesis (i) above, and it is strictly less than γ on the complementary arch. This means that if we deform the ζ -integration contour so that it goes slightly inside the unit circle for $\arg \zeta \in (-c(\gamma), c(\gamma))$, and slightly outside the unit circle for $|\arg \zeta| > c(\gamma)$, then the factor $F(\tau, \zeta) \zeta^{-x-\frac{1}{2}}$ with $x \sim \gamma/|\ln s|$ will exponentially decay on this contour.

Similarly, for $\tau = N - \hat{\tau}$ with nonnegative and bounded $\hat{\tau}$ we have

$$F(\tau, \zeta) = \prod_{k=0}^{\hat{\tau}} (1 - s^k \zeta / t) \frac{\prod_{k=1}^{N/2} (ts^k / \zeta; t)_{\infty}}{\prod_{k=0}^{N/2-1} (s^k \zeta / t; t)_{\infty}}$$

which has the same asymptotically nontrivial part as before up to the change $\zeta \rightarrow t\zeta$. This means that in this case the desired ζ -integration contour is obtained from the one in the previous case by multiplication by t .

The proof of (i) proceeds in the same way as the proof of Proposition 7.1: The deformation of contours to the desired ones meets an obstacle of the theta function $\theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)$ in the denominator of the integrand vanishing for $\zeta\eta = 1$ or $\zeta\eta = t$; evaluating the corresponding residue yields the limit correlation kernel, while the remaining integrals over the constructed contours tend to zero as $s \rightarrow 1$. The estimates are uniform in z , $|z| = 1$, and the limiting kernel is independent of z ; thus, the integration over z in the formula for the correlation functions in Proposition 5.2 can be removed.

In order to prove (ii) we need to consider $F(\tau, \zeta)$ where $\tau = N/2 + \tilde{\tau}$ with bounded $\tilde{\tau}$. The asymptotically nontrivial part is independent of the sign of $\tilde{\tau}$, so let us take $\tilde{\tau} > 0$ to be concrete. We have

$$F(\tau, \zeta) = \prod_{k=1}^{\tilde{\tau}} (1 - \sqrt{t} s^k \zeta) \frac{\prod_{k=1}^{N/2} (ts^k/\zeta; t)_{\infty}}{\prod_{k=0}^{N/2-1} (s^k \zeta; t)_{\infty}}.$$

The first factor has a finite limit as $s \rightarrow 1$, while the second factor (the ratio) produces

$$\exp \frac{1}{|\ln s|} \sum_{m=0}^{\infty} (\operatorname{dilog}(\zeta t^m) - \operatorname{dilog}(\zeta t^m \sqrt{t}) - \operatorname{dilog}(t^{m+1}/\zeta) + \operatorname{dilog}(t^{m+1} \sqrt{t}/\zeta)).$$

The real part of this sum vanishes on the circle $|\zeta| = \sqrt{t}$. Further, on this circle the derivative of the sum multiplied by ζ equals

$$2 \ln \left| \frac{(\zeta_0 t; t)_{\infty}}{(\zeta_0 \sqrt{t}; t)_{\infty}} \right|, \quad \zeta_0 = \frac{\zeta}{|\zeta|}.$$

The remaining part of the proof is just as in the case (i) considered above. It is worth noting that for $\gamma > \gamma_2^{\max}$ the deformation of the contours to the domain of exponential decay of the integrand does not meet any obstacles, and thus the limiting kernel is identically equal to zero. \square

8. ON A MEASURE OF NEKRASOV AND OKOUNKOV

A large part of the material presented in this section is the result of joint discussions of the author and Grigori Olshanski.

Our goal in this final section is to demonstrate how the techniques developed in the first three sections apply to a remarkable measure on partitions introduced by Nekrasov and Okounkov in [NO].

Fix $\mu \in \mathbb{C}$ and $t \in (0, 1)$. The object of interest is a (generally speaking, complex) measure on the set of all partitions given by the formula, see [NO, §6.2],

$$M_{\mu, t}(\lambda) = \prod_{n \geq 1} (1 - t^n)^{1 - \mu^2} \cdot t^{|\lambda|} \prod_{\square \in \lambda} \frac{h(\square)^2 - \mu^2}{h(\square)^2}, \quad \lambda \in \mathbb{Y}.$$

Here the product is taken over all boxes of the Young diagram λ , and $h(\square)$ denotes the length of the hook rooted at the given box. The sum of the weights $M_{\mu,t}(\lambda)$ over all partitions λ is identically equal to 1, see [NO, (6.12)]. Note that this measure becomes a *probability* measure (meaning that all weights are nonnegative) if $\mu \in i\mathbb{R}$.

One interesting feature of measures $M_{\mu,t}$ is that they interpolate between the uniform measure on partitions arising at $\mu = 0$, and the poissonized Plancherel measure which is obtained by the limit transition

$$\lim_{\substack{\mu \rightarrow \infty, t \rightarrow 0 \\ \mu^2 t \rightarrow -\theta}} M_{\mu,t}(\lambda) = e^{-\theta|\lambda|} \prod_{\square \in \lambda} \frac{1}{h(\square)^2} = e^{-\theta} \left(\frac{\dim \lambda \theta^{\frac{|\lambda|}{2}}}{|\lambda|!} \right)^2.$$

Here $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$. We refer the reader to [BOO] for details and further references on the Plancherel measures.

Let us denote by ρ_μ the specialization of the algebra of symmetric functions Λ such that

$$h_n(\rho_\mu) = \frac{(\mu)_n}{n!}, \quad n \geq 0, \quad \text{or} \quad H(\rho_\mu, u) = (1 - u)^{-\mu}.$$

Here $(a)_k = a(a+1) \dots (a+k-1)$ denotes the Pochhammer symbol.

The applicability of the periodic Schur process results follows from

Lemma 8.1. *For any $\kappa, \lambda \in \mathbb{Y}$ choose $r \geq \max\{\ell(\kappa), \ell(\lambda)\}$ and set*

$$k_i = \kappa_i - i + r, \quad l_i = \lambda_i - i + r, \quad i = 1, \dots, r.$$

Then

$$\begin{aligned} \sum_{\nu \in \mathbb{Y}} s_{\kappa/\nu}(\rho_\mu) s_{\lambda/\nu}(\rho_{-\mu}) &= \prod_{\square \in \kappa} \frac{h(\square) + \mu}{h(\square)} \prod_{\square \in \lambda} \frac{h(\square) - \mu}{h(\square)} \\ &\quad \times \frac{(-1)^{\frac{r(r-1)}{2}} \mu^r \prod_{1 \leq i < j \leq r} (k_i - k_j + \mu)(l_i - l_j - \mu)}{\prod_{i,j=1}^r (\mu + k_i - l_j)}. \end{aligned}$$

In particular, for $\kappa = \lambda$ the last factor turns into 1 and we obtain

$$\sum_{\nu \in \mathbb{Y}} s_{\lambda/\nu}(\rho_\mu) s_{\lambda/\nu}(\rho_{-\mu}) = \prod_{\square \in \lambda} \frac{h(\square)^2 - \mu^2}{h(\square)^2}.$$

Comments. 1. Only finitely many terms in the sums over ν above are nonzero.

2. The formula for $\kappa = \lambda$ can be easily extracted from the Fock space representation of the measure $M_{\mu,t}$ given in [NO].

Proof. Since the terms of the sum vanish unless $\nu \subset \kappa$ and $\nu \subset \lambda$, we can restrict the sum to ν with $\ell(\nu) \leq r$. Denote $n_i = \nu_i - i$, $i = 1, \dots, r$.

The Jacobi-Trudi formula for the skew Schur functions, see [Macd, §I (5.4)], gives

$$s_{\kappa/\nu} = \det[h_{k_i - n_j}]_{i,j=1}^r, \quad s_{\lambda/\nu} = \det[h_{l_i - n_j}]_{i,j=1}^r.$$

Applying the Cauchy-Binet summation formula, we obtain

$$\begin{aligned} \sum_{\nu \in \mathbb{Y}} s_{\kappa/\nu}(\rho_\mu) s_{\lambda/\nu}(\rho_{-\mu}) &= \sum_{n_1 > \dots > n_r \geq 0} \det[h_{k_i - n_j}(\rho_\mu)]_{i,j=1}^r \det[h_{l_i - n_j}(\rho_{-\mu})]_{i,j=1}^r \\ &= \det \left[\sum_{m=0}^{\min\{k_i, l_j\}} h_{k_i - m}(\rho_\mu) h_{l_j - m}(\rho_{-\mu}) \right]_{i,j=1}^r = \det \left[\sum_{m=0}^{\min\{k_i, l_j\}} \frac{(\mu)_{k_i - m}}{(k_i - m)!} \frac{(\mu)_{l_j - m}}{(l_j - m)!} \right]_{i,j=1}^r \end{aligned}$$

Using the identity

$$\sum_{m=0}^{\min\{k,l\}} \frac{(\mu)_{k-m}}{(k-m)!} \frac{(\mu)_{l-m}}{(l-m)!} = \frac{\mu(\mu+1)_k (-\mu+1)_l}{k! l! (\mu+k-l)},$$

which can be proved by induction on $\min\{k, l\}$ with $k-l$ fixed, and the formula for the Cauchy determinant

$$\det \left[\frac{1}{\mu + k_i - l_j} \right]_{i,j=1}^r = (-1)^{\frac{r(r-1)}{2}} \frac{\prod_{1 \leq i < j \leq r} (k_i - k_j)(l_i - l_j)}{\prod_{i,j=1}^r (\mu + k_i - l_j)},$$

we obtain

$$\begin{aligned} \sum_{\nu \in \mathbb{Y}} s_{\kappa/\nu}(\rho_\mu) s_{\lambda/\nu}(\rho_{-\mu}) &= (-1)^{\frac{r(r-1)}{2}} \mu^r \prod_{i=1}^r \frac{(\mu+1)_{k_i} (-\mu+1)_{l_i}}{k_i! l_i!} \frac{\prod_{1 \leq i < j \leq r} (k_i - k_j)(l_i - l_j)}{\prod_{i,j=1}^r (\mu + k_i - l_j)}. \end{aligned}$$

Finally, we use the fact that the set of hook lengths $\{h(\square)\}_{\square \in \kappa}$ can be obtained as the union of sets $\{1, \dots, k_i\}$ for $i = 1, \dots, r$ minus the set of numbers $\{k_i - k_j\}_{1 \leq i < j \leq r}$, and similarly for λ . This brings us to the desired formula. \square

Let us now associate to the measure $M_{\mu,t}$ a measure on point configurations (subsets) in $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ via the map

$$\mathbb{Y} \longrightarrow 2^{\mathbb{Z}'}, \quad \lambda \mapsto \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1}.$$

The correlation functions of this measure on point configurations are given by

$$\rho_n(x_1, \dots, x_n) = \sum_{\lambda \in \mathbb{Y}: \{\lambda_i - i + \frac{1}{2}\}_{i \geq 1} \supset \{x_1, \dots, x_n\}} M_{\mu,t}(\lambda).$$

We will also need the *shift-mixed* version of $M_{\mu,t}$. For any $z \in \mathbb{C} \setminus \{-t^{\pm \frac{1}{2}}, -t^{\pm \frac{3}{2}}, \dots\}$ define the measure $M_{\mu,t,z}$ on $\mathbb{Y} \times \mathbb{Z}$ as the product-measure

$$M_{\mu,t,z}(\lambda, S) = \frac{z^S t^{\frac{S^2}{2}}}{\theta_3(z; t)} M_{\mu,t}(\lambda),$$

cf. §2. We associate to this measure a measure on point configurations in \mathbb{Z}' via the map

$$\mathbb{Y} \times \mathbb{Z} \longrightarrow 2^{\mathbb{Z}'}, \quad \lambda \mapsto \{S + \lambda_i - i + \frac{1}{2}\}_{i \geq 1}.$$

Its correlation functions are given by

$$\rho_n^{\text{shift}}(x_1, \dots, x_n) = \sum_{(\lambda, S) \in \mathbb{Y} \times \mathbb{Z}: \{S + \lambda_i - i + \frac{1}{2}\}_{i \geq 1} \supset \{x_1, \dots, x_n\}} M_{\mu,t,z}(\lambda, S).$$

The two sets of correlation functions are related as described in Proposition 2.1.

Proposition 8.2. *The correlation functions of the shift-mixed measure $M_{\mu,t,z}$ have determinantal form: For any $n = 1, 2, \dots$ and $x_1, \dots, x_n \in \mathbb{Z}'$,*

$$\rho_n^{\text{shift}}(x_1, \dots, x_n) = \det[K_{\mu,t,z}(x_i, x_j)]_{i,j=1}^n,$$

where the correlation kernel has the form

$$K_{\mu,t,z}(x, y) = -\frac{\prod_{n \geq 1} (1 - t^n)^3}{\theta_3(z; t) (2\pi i)^2} \times \oint_{\zeta} \oint_{\eta} \prod_{m \geq 0} \frac{(1 - \zeta t^m)^\mu (1 - t^{m+1}/\zeta)^\mu}{(1 - t^m/\eta)^\mu (1 - \eta t^{m+1})^\mu} \frac{\theta_3(z\zeta\eta; t)}{\theta_3(-\zeta\eta t^{-\frac{1}{2}}; t)} \frac{d\zeta d\eta}{\zeta^{x+\frac{1}{2}} \eta^{y+\frac{1}{2}}}.$$

Here both integration contours are simple positively oriented loops going around the origin such that

$$t < |\zeta| < 1 < |\eta| < t^{-1}, \quad 1 < |\zeta\eta| < t^{-1},$$

and we use the principal branch of the logarithm to define $(\cdot)^\mu$.

Proof. Consider the periodic Schur process with $N = 1$ and specializations $a = a[1]$ and $b = b[1]$ of Λ defined by

$$H(a; u) = (1 - \alpha u)^{-\mu}, \quad H(b; u) = (1 - \alpha u)^\mu, \quad 0 < \alpha < 1.$$

Then by Proposition 1.1

$$\sum_{\lambda \in \mathbb{Y}} t^{|\lambda|} \sum_{\nu \in \mathbb{Y}} s_{\lambda/\nu}(a) s_{\lambda/\nu}(b) = \prod_{n \geq 1} \frac{(1 - \alpha^2 t^n)^{\mu^2}}{1 - t^n},$$

which converges to $\prod_{n \geq 1} (1 - t^n)^{\mu^2 - 1}$ as $\alpha \rightarrow 1$. Since $f(a) \rightarrow f(\rho_\mu)$ and $f(b) \rightarrow f(\rho_\mu)$ for any $f \in \Lambda$ as $\alpha \rightarrow 1$, we see that the weights of the periodic Schur process above as well as those of its shift-mixed version converge to the corresponding weights of $M_{\mu,t}$ and $M_{\mu,t,z}$. Then the fact that the sum of the weights in all cases is identically equal to one, implies the convergence of the correlation functions of the (shift-mixed) Schur process to those of $M_{\mu,t}$ and $M_{\mu,t,z}$.

On the other hand, the analytic version of Theorem 2.2 (see also Remarks 2.3 and 2.4) gives the formula for the correlation functions of the shift-mixed Schur process introduced above. Taking the limit $\alpha \rightarrow 1$ in the integral representation of the kernel given in Remark 2.4 completes the proof. \square

Let us now restrict our attention to the case when the measures $M_{\mu,t}$ and $M_{\mu,t,z}$ become probability measures (in other words, all weights are nonnegative). This happens when $\mu = i\mu_0$ with $\mu_0 \in \mathbb{R}$, and $z \in \mathbb{R}_+$.

Theorem 8.3. *Assume that $\mu = i\mu_0$ with $\mu_0 \in \mathbb{R}$, and $z > 0$. Then as $t \rightarrow 1$ the correlation functions of the shift-mixed measure $M_{\mu,t,z}$ have the following limit: Choose $x_1(t), \dots, x_n(t) \in \mathbb{Z}'$ such that as $t \rightarrow 1$, $|\ln t| \cdot x_k(t) \rightarrow \gamma$ for all $k = 1, \dots, n$ and some $\gamma \in \mathbb{R}$, and all pairwise distances $x_i - x_j = x_i(t) - x_j(t)$ are independent of t . Then*

$$\lim_{t \rightarrow 1} \rho_n^{\text{shift}}(x_1(t), \dots, x_n(t)) = \det[\mathcal{K}^{(z, \gamma, \mu)}(x_i - x_j)]_{i,j=1}^n,$$

where the correlation kernel has the following form

$$\mathcal{K}^{(z, \gamma, \mu)}(d) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{1}{1 + z^{-1}e^\gamma(1-\zeta)^{-\mu}(1-\zeta^{-1})^\mu} \frac{d\zeta}{\zeta^{d+1}}, \quad d \in \mathbb{Z}.$$

Under the same assumptions the correlation functions of the measure $M_{\mu, t}$ converge to the limiting expression above evaluated at $z = 1$.

Comments. 1. For $|\zeta| = 1$ we have

$$(1 - \zeta)^{-i\mu_0}(1 - \zeta^{-1})^{i\mu_0} = \begin{cases} e^{\mu_0(\arg \zeta - \pi)}, & 0 < \arg \zeta \leq \pi, \\ e^{\mu_0(\arg \zeta + \pi)}, & -\pi \leq \arg \zeta < 0. \end{cases}$$

The change of sign of μ_0 is equivalent to the change of variable $\zeta \rightarrow \zeta^{-1}$ in the integral above, which in its turn is equivalent to transposing the correlation kernel. Clearly, this operation does not change the correlation functions.

2. The global limit density function for the shift-mixed case is equal to

$$\begin{aligned} \rho(\gamma) = \mathcal{K}^{(z, \gamma, \mu)}(0) &= \frac{1}{2\pi} \left(\int_0^\pi \frac{d\phi}{1 + z^{-1}e^{\gamma + \mu_0(\phi - \pi)}} + \int_{-\pi}^0 \frac{d\phi}{1 + z^{-1}e^{\gamma + \mu_0(\phi + \pi)}} \right) \\ &= \frac{1}{2\pi\mu_0} \ln \frac{e^\gamma + ze^{\pi\mu_0}}{e^\gamma + ze^{-\pi\mu_0}}, \end{aligned}$$

and one has to substitute $z = 1$ to get the formula corresponding to the non-mixed measure $M_{\mu, t}$.

3. In the limit $\mu_0 \rightarrow +\infty$, $\gamma \rightarrow \infty$ so that $\gamma/\mu_0 \rightarrow \hat{\gamma}$, the correlation kernel $\mathcal{K}^{(z, \gamma, \mu)}$ becomes equivalent to the discrete sine kernel, cf. Example 3.4:

$$\lim_{\substack{\mu_0 \rightarrow +\infty, \gamma \rightarrow \infty, \\ \gamma/\mu_0 \rightarrow \hat{\gamma}}} \mathcal{K}^{(z, \gamma, \mu)}(x - y) = \begin{cases} 0, & \hat{\gamma} \geq \pi, \\ e^{\frac{i}{2}(\pi - \hat{\gamma})(x - y)} \cdot \frac{\sin\left(\frac{\pi - \hat{\gamma}}{2}(x - y)\right)}{\pi(x - y)}, & -\pi < \hat{\gamma} < \pi, \\ \delta(x - y), & \hat{\gamma} \leq -\pi. \end{cases}$$

The proof of Theorem 8.3 is completely analogous to that of Theorem 3.1 and we omit it.

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